# Gauge invariant non-linear electrodynamics motivated by a spontaneous breaking of the Lorentz symmetry

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We introduce a new version of non-linear electrodynamics which is produced by a spontaneous symmetry breaking of Lorentz invariance induced by the non-zero expectation value of the gauge invariant electromagnetic field strength. The symmetry breaking potential is argued to effectively arise from the integration of massive gauge bosons and fermions in an underlying fundamental theory. All possible choices of the vacuum lead only to the remaining invariant subgroups T(2) and HOM(2). We explore in detail the plane wave solutions of the linearized sector of the model for an arbitrary vacuum. They present two types of dispersion relations. One corresponds to the case of the usual Maxwell electrodynamics with the standard polarization properties of the fields. The other dispersion relation involves anisotropies determined by the structure of the vacuum. The corresponding fields reflect these anisotropies. The model is stable in the small Lorentz invariance violation approximation. We have also embedded our model in the photon sector of the Standard Model Extension, in order to translate the many bounds obtained in the latter into corresponding limits for our parameters. The one-way anisotropic speed of light is calculated for a general vacuum and its isotropic component is strongly bounded by $\tilde{\delta}c/c < 2 \times 10^{-32}$. The anisotropic violation contribution is estimated by introducing an alternative definition for the difference of the two-way speed of light in perpendicular directions, $\Delta c$, that is relevant to Michelson-Morley type of experiments and which turns out to be also strongly bounded by $\Delta c/c < 10^{-32}$. Finally, we speculate on the relation of the vacuum energy of the model with the cosmological constant and propose a connection between the vacuum fields and the intergalactic magnetic fields.

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## I. INTRODUCTION

The possible violation of Lorentz invariance has recently received a lot of attention, both from the experimental and theoretical sides. In the latter case, mainly in connection with possible effects arising from drastic modifications of space-time at distances of the order of Planck length suggested by most of the current quantum gravity approaches. Experiments and astrophysical observations set stringent bounds upon the parameters describing such violations, which nevertheless are still been subject to improvement in their precision. Broadly speaking, there are two basic possibilities which produce such a breaking: (1) one introduces by hand a number of non-dynamical tensor fields, whose fixed directions induce the corresponding breaking in a given reference frame. Examples of this approach are the phenomenological Myers-Pospelov model [1] together with QED in a constant axial vector background [2]. (2) a second possibility is to dwell upon spontaneous symmetry breaking (SSB). This is basically what is done in the Standard Model Extension [3], [4], where such non-dynamical tensor fields are assumed to arise from vacuum expectation values of some basic fields belonging to a more fundamental model, like string theory for example.

Nevertheless, we are interested in studying a model of electrodynamics which incorporates spontaneous Lorentz symmetry breaking, without having to go to a string theory setting. In order to produce such a model we require the presence of the non-zero vacuum expectation value of a tensorial field of rank greater than zero.

A first possibility arises from having a non-zero VEV of the potential field $A_\mu$. This case has been thoroughly studied and leads to the interesting idea that the photon arises as the corresponding Goldstone boson of the global spontaneous Lorentz symmetry breaking. Normally the masslessness of the photon is explained by invoking abelian gauge invariance under the group $U(1)$. This almost sacred principle has undoubtedly been fundamental in the development of physics and it is very interesting to explore the possibility that it could have a dynamical origin [5]. This idea goes back to the works of Nambu [6] and Bjorken [7] together with many other contributions [8, 9]. Recently it has been revived in references [10, 11, 12, 13, 14]. One of the most explored approaches along these lines starts form a theory with a vector field $B^\mu$ endowed with the standard electrodynamics kinetic term plus a potential designed to break Lorentz symmetry via a non-zero vacuum expectation value $< B^\mu >$, which defines a preferred direction in space-time. This potential also breaks gauge invariance. These are the so called bumblebee models [15]. The subsequent symmetry breaking, obtained from the non-zero minimum of the potential, splits the original four degrees of freedom in $B^\mu$ into three vectorial Nambu-Goldstone bosons $A^\mu$, satisfying the constraint $A_\mu A^\mu = \pm M^2$, to be identified with the photon, plus a massive scalar field $\sigma$, which is assumed to be excited at very high energies. Under this threshold one basically recovers the Lagrangian for electrodynamics in vacuum plus the above mentioned

non-linear constraint, which is interpreted as a gauge fixing condition. Calculations at the tree level [6] and to the one loop level [12] have been carried on, showing that the possible Lorentz violating effects are not present in the physical observables and that the results are completely consistent with the standard gauge invariant electrodynamics. This is certainly a surprising result, which can be really appreciated from the complexity of the calculation in Ref. [12]. Nevertheless, alternative kinetic terms in the bumblebee models can lead to a theory differing substantially from electromagnetism [16]. The use of nonpolynomial interactions within this framework has been explored in [17]. The idea of a photon as a Goldstone boson has also been extended to gravitons in general relativity [15, 18, 19].

It is noteworthy to recall here that the constraint $A_\mu A^\mu = \pm M^2$ was originally proposed by Dirac as a way to derive the electromagnetic current from the additional excitations of the photon field, which now ceased to be gauge degrees of freedom, avoiding in this way the problems arising from considering point-like charges [20]. As reported in [21], this theory requires the existence of an ether emerging from quantum fluctuations, but not necessarily of the violation of Lorentz invariance, due to an averaging process of the quantum states producing such fluctuations.

In this work we explore the possibility of having a non-zero VEV of the electromagnetic strength, so that we avoid the problem of disturbing gauge invariance from the very beginning. To this end we consider a non-linear version of Electrodynamics with a potential term having an stable minimum for a constant electromagnetic strength. The paper is organized as follows: In Section II we make it plausible the existence of such gauge invariant potential $V_{eff}(F^2)$, arising as an effective low energy contribution from the integration of degrees of freedom corresponding to massive gauge-bosons and fermions in a more fundamental theory. This potential also exhibits the right behavior in the low-intensity field approximation. Section III contains the construction of the model starting from a non-linear version of Electrodynamics which is spontaneously broken by choosing a vacuum characterized by a constant electromagnetic tensor $C_{\mu\nu}$. In Section IV we discuss the possible realizations of such vacuum in terms of the associated electromagnetic fields $\mathbf{e}$ and $\mathbf{b}$, identifying the Lorentz subgroups that remain invariant after the breaking. The equations of motion corresponding to the propagating sector, i.e. those arising from the quadratic terms in the Lagrangian, are examined in Section V using a covariant approach, as well as the standard $3+1$ decomposition. Also, the modified Maxwell equations are presented. Section VI contains a summary of the modified photon dispersion relations $\omega = \omega(\mathbf{k})$, together with the expressions for the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ of plane waves propagating in the different vacua parameterized by $\mathbf{e}$ and $\mathbf{b}$. The model is shown to be stable in the small Lorentz invariance violation approximation. In section VII we study the embedding of our model in the corresponding sector of the Standard Model Extension [24] and use the experimentally established bounds upon the LIV parameters to constrain the electromagnetic fields $\mathbf{e}$ and $\mathbf{b}$. Finally we conclude with a summary of the work together with some comments in Section VIII. The Appendix contains some details of the calculation leading to the motivation described in Section II.

## II. MOTIVATION

One of the main challenges of the proposed approach is to make it plausible the existence of a gauge invariant potential $V(F_{\alpha\beta})$ depending upon the electromagnetic tensor, which is stable for large values of the electric and magnetic fields and which possess a minimum for some constant value of $F_{\alpha\beta}$. In particular, we will consider at this level only the case of magnetic fields in order to avoid further instabilities due to pair creation in strong electric fields.

A first possibility that arises is to consider the effective photon interaction in QED after integrating the fermions. Unfortunately, a detailed calculation of this potential, in the one-loop $\left(V^{(1)}_{EM}\right)$ and two-loop $\left(V^{(2)}_{EM}\right)$ approximations [25], shows that it is given by

$$V_{EM}(B) = \frac{1}{2}B^2 + V^{(1)}_{EM} + V^{(2)}_{EM} = \frac{1}{2}B^2 - \frac{e^2}{24\pi^2}B^2\left[\ln\left(\frac{eB}{m^2}\right)\right] - \frac{e^4}{128\pi^4}B^2\ln\left(\frac{eB}{m^2}\right), \tag{1}$$

in the limit when $B >> B_0 = m^2/e$, which clearly is unbounded from below.

A new possibility arises in a further generalization of the mechanism proposed in Refs. [7, 8], restricted to the gauge invariant case. We start from a conventional gauge theory including fermions, gauge fields and Higgs fields which provide masses, via spontaneous symmetry breaking, to the gauge fields, except for the photon potential $A_\mu$, which carries the electromagnetic $U(1)$ gauge invariance. For simplicity we consider only one fermion species, as in Ref.[7], but the idea can be generalized to many of them. In the latter case, the subsequent integration over the fermionic degrees of freedom could be made only for those having a mass larger than a given scale, thus leaving unintegrated the lightest ones.

Integrating the massive gauge bosons of this theory leads to a power series in bilinears in the fermion fields, which are suppressed by powers of the energy scale $\Lambda$ that characterizes the SSB mechanism providing the mass to the gauge vector bosons [8]. We focus upon the neutral sector of the model where the following contributions to the effective

Lagrangian arise

$$\frac{1}{\Lambda^{6n-4}}\left[\sum_a \left(\bar{\Psi}M_a\Psi\right)\left(\bar{\Psi}M^a\Psi\right)\right]^n, \quad \frac{1}{\Lambda^{8n-4}}\left[\sum_a \left(\bar{\Psi}M_a D_\nu\Psi\right)\left(\bar{\Psi}M^a D^\nu\Psi\right)\right]^n, \ldots, \tag{2}$$

We are considering all possible contributions which are consistent with gauge invariance, the remaining symmetry of this sector of the theory. Notice also that terms containing the $U(1)$ covariant derivative $D_\nu = (\partial_\nu + ieA_\nu)$ are further suppressed with respect to those without it. Also we consider only the $n = 1$ case, which provides a gauge invariant generalization of the model in Ref. [7]. Thus our starting point is

$$L_{eff} = \bar{\Psi}\left(i\gamma^\mu\left(\partial_\mu + ie\tilde{A}_\mu\right) - m\right)\Psi - \sum_{M,a}\frac{r_M}{\Lambda^2}\left[\left(\bar{\Psi}M_a\Psi\right)\left(\bar{\Psi}M^a\Psi\right)\right]. \tag{3}$$

The index $M : S, V, T, PV, PS$ labels the tensorial objects that constitute the Dirac basis: Scalar, Vector, Tensor, Pseudovector and Pseudoscalar, respectively. The generic index $a$ labels the covariant (contravariant) components of each tensor class. More details are given in Table I of the Appendix. We follow the Dirac algebra conventions of Ref. [22], with appropriate factors chosen in such a way that each current $\left(\bar{\Psi}M^a\Psi\right)$ is real. In this way, the proposed generalization includes all possible fermionic quartic interactions. Also, the coefficients $r_M$ are assumed to be of order one.

The fermions can be integrated next by introducing the corresponding auxiliary fields $(C_M)^a$ for each real current $(j_M)_a = \bar{\Psi}M_a\Psi$, as suggested in Ref. [8], and following standard manipulations in the path integral formulation. Some details are given in the Appendix.

Our main goal is to estimate the additional corrections provided by such currents to the purely electromagnetic effective potential (1) and to investigate whether or not it is possible to generate a net positive contribution in the high-intensity field limit. To have a preliminary simpler expression for the resulting extension we choose to consider separately each of the contributions arising from the different currents and only add them to the total effective potential at the end. In this way we are nor considering possible interference terms among the currents, which nevertheless could be calculated within the approximations performed. Calling $V^{(2)}$ the finite part of the total effective potential arising from the currents $j_M$, given in Eq.(259) of the Appendix, we obtain

$$B \;\to\; \infty: \quad V^{(2)} = \left(\frac{1}{16\pi^2}\right)^2\left(\frac{m}{\Lambda}\right)^2 e^2B^2\left[\ln\left(\frac{2eB}{m^2}\right)\right]^2\sum_M r_M\,\theta_M, \tag{4}$$

$$B \;\to\; 0: \quad V^{(2)} = \left(\frac{1}{16\pi^2}\right)^2\left(\frac{m}{\Lambda}\right)^2\left(\frac{eB^2}{3m^2} + \frac{1}{18}\frac{e^2B^4}{m^4}\right)^2\sum_M r_M\,\theta_M. \tag{5}$$

Here $m$ is the mass of the integrated fermion which is such that $m > \Lambda$. The main point of the estimation is that we obtain

$$\Theta = \sum_M r_M\,\theta_M = 4r_S + 16r_V + 48r_T - 16r_{PV} - 4r_{PS}, \tag{6}$$

which can be made positive from a judicious choice of the parameters $r_M$.

In the high-intensity field approximation, the dominant term in the electromagnetic contribution goes like

$$V_{EM}(B) \simeq -e^2B^2\ln\left(\frac{eB}{m^2}\right), \tag{7}$$

which can be overcame by the contribution from the additional currents

$$V^{(2)} \simeq \left(\frac{m}{\Lambda}\right)^2 e^2B^2\left[\ln\left(\frac{2eB}{m^2}\right)\right]^2\Theta, \tag{8}$$

provided $\Theta > 0$.

As we mention in the Appendix, this preliminary estimation has only considered an appropriate regularization of the divergent integrals, but is missing an adequate renormalization of the total effective Lagrangian $L_{eff}$. Our renormalization conditions would be

$$\lim_{B\to 0} L_{eff} = \rho B^2, \qquad \rho > 0, \tag{9}$$

in such a way that the effective potential is a decreasing function in the vicinity of $B = 0$. The final normalization to the value of $L_{eff} = -b^2/2$ will be imposed at the end of the procedure, after we have expanded the magnetic field around an stable minimum $B_{Min}$ of $V_{eff}$, such that

$$B = B_{Min} + b \tag{10}$$

and we are left with the physical component $b$. The renormalization procedure should be analogous to the one carried in Ref. [25] for the two-loop contribution $L^{(2)}_{EM}(A)$ to the effective electromagnetic action and it is rather involved due to the complicated field dependence of the regularized version of the divergent integrals. This calculation is beyond the scope of these preliminary estimations.

Having an effective potential with the characteristics described above guarantees the existence of an absolute minimum, which we assume to arise at values lower that the critical magnetic and electric fields, in such a way to avoid fermion pair production in the later case. We also generalize this idea to the case of an arbitrary constant electromagnetic field $F_{\mu\nu}$. The study of the stability of the model under radiative corrections is beyond the scope of the present work.

## III. THE MODEL

In the previous section we have made it plausible the existence of a gauge invariant potential $V_{eff}(F^2)$ having an stable minimum and arising as an effective low energy model from the integration of degrees of freedom corresponding to massive gauge-bosons and fermions in a more fundamental theory. This potential also exhibits the right behavior in the low-intensity field approximation. To examine the dynamical consequences of the symmetry breaking we have to adopt a convenient parametrization of such potential. There are various choices in the literature which would translate into the following for our case [23]

$$V(F) = \lambda(F^2 \pm C^2), \qquad V(F) = \frac{\lambda}{4}(F^2 \pm C^2)^2, \tag{11}$$

where $C_{\mu\nu}$ is the constant value of the electromagnetic field at the minimum. Instead we will adopt the standard Ginzburg-Landau parametrization

$$V(F_{\mu\nu}) = \frac{1}{2}\alpha F^2 + \frac{\beta}{4}\left(F^2\right)^2 . \tag{12}$$

Since the basic variable we want to deal with is $F_{\mu\nu}$, we start from the effective Lagrangian

$$L(F_{\alpha\beta}, X_\mu) = -V(F_{\alpha\beta}) - \bar{F}^{\nu\mu}\partial_\nu X_\mu, \quad F_{\alpha\beta} = -F_{\beta\alpha}, \tag{13}$$

where the dual field $\bar{F}^{\nu\mu}$ is

$$\bar{F}^{\nu\mu} = \frac{1}{2}\epsilon^{\nu\mu\alpha\beta}F_{\alpha\beta} \tag{14}$$

and the $X_\mu$ are just Lagrange multipliers that will finally impose the condition that the field strength satisfies the Bianchi identity and can then be expressed in terms of the vector potential. The above Lagrangian is similar to that used in the standard formulation of non-linear electrodynamics [26]. Our conventions are $\eta_{\mu\nu} = diag(+,-,-,-), \quad \epsilon^{0123} = +1, \quad \epsilon_{123} = +1$.

The next step is to determine the vacuum configuration of the theory, corresponding to the minimum of the energy $E$. The energy density $T^{00}$ is calculated via the standard Noether theorem starting from the Lagrangian (13) and produces

$$E = \int d^3x\, T^{00} = \int d^3x\, \left[V(F_{\alpha\beta}) + \bar{F}^{i\alpha}\left(\partial_i X_\alpha\right)\right] . \tag{15}$$

The extremum conditions, obtained by varying the independent fields $F_{\alpha\beta}$ and $X_\alpha$, are

$$\frac{\delta E}{\delta F_{\rho\sigma}} = \frac{\partial V}{\partial F_{\rho\sigma}} + \left(\partial_i X_\alpha\right)\epsilon^{i\alpha\rho\sigma} = 0, \tag{16}$$

$$\frac{\delta E}{\delta X_\alpha} = -\frac{1}{2}\epsilon^{i\alpha\rho\sigma}\partial_i F_{\rho\sigma} = 0. \tag{17}$$

In order to maintain four dimensional translational invariance we look for extrema of the fields which are independent of the coordinates. From Eqs. (16) and (17) we verify that this is possible provided the condition

$$\left(\frac{\partial V}{\partial F_{\rho\sigma}}\right)_{Extr.} = 0. \tag{18}$$

holds. In other words, to find the vacuum configuration we have to extremize the effective action, subjected to the condition that $F_{\alpha\beta}$ and $X_\alpha$ are constant fields. Applying these requirements to (13) plus the choice (12) we obtain

$$\frac{\partial V}{\partial F^{\mu\nu}} = 0 = \left(\alpha + \beta F^2\right) F_{\mu\nu}, \tag{19}$$

which is solved by a constant $(F_{Extr})_{\alpha\beta} \equiv C_{\alpha\beta}$ such that

$$\left(F^2\right)_E = -\frac{\alpha}{\beta} = C^2 \neq 0. \tag{20}$$

The expansion around the minimum $(C_{\mu\nu},\ C_\mu)$ is written as

$$F_{\alpha\beta}(x) = C_{\alpha\beta} + a_{\alpha\beta}(x), \qquad X_\mu = C_\mu + \bar{X}_\mu. \tag{21}$$

Introducing such expansion in the potential (12), we arrive to

$$V(F_{\alpha\beta}) = V(C_{\alpha\beta}) + \beta\left(C_{\alpha\beta}a^{\alpha\gamma}\right)^2 + \beta\left(C_{\alpha\beta}a^{\alpha\beta}\right)\left(a_{\mu\nu}a^{\mu\nu}\right) + \frac{\beta}{4}\left(a_{\alpha\beta}a^{\alpha\beta}\right)^2, \tag{22}$$

$$\equiv V(C_{\alpha\beta}) + \bar{V}(a_{\alpha\beta}). \tag{23}$$

We notice that (22) has no linear term in $a^{\alpha\gamma}$, and also that the quadratic term $\beta\left(C_{\alpha\beta}a^{\alpha\gamma}\right)^2$ is positive provided

$$\beta > 0, \tag{24}$$

thus verifying the expansion around a minimum. On the other hand, according to Eq. (20), the sign of $\alpha$ is arbitrary, in such a way that it is opposite to the sign of $C^2$.

In this way, the spontaneously symmetry broken action is

$$S(a_{\alpha\beta}, \bar{X}_\mu\ ) = -\int d^4x \left(\frac{1}{2}\epsilon^{\nu\mu\alpha\beta}a_{\alpha\beta}\partial_\nu\bar{X}_\mu + \bar{V}(a_{\alpha\beta}) + V(C)\right), \tag{25}$$

where we have eliminated the total derivatives arising from the shifts of the fields (21) performed in the original Lagrangian (13).

Next we consider the equations of motion derived from (25) and show that the Lagrange multiplier $\bar{X}_\mu$ is fully determined up to gauge transformation $\bar{X}_\mu \longrightarrow \bar{X}_\mu + \partial_\mu\chi$. The equations are

$$\delta a_{\alpha\beta}\ :\ -\epsilon^{\nu\mu\alpha\beta}\partial_\nu\bar{X}_\mu - 2\frac{\partial\bar{V}}{\partial a_{\alpha\beta}} = 0, \tag{26}$$

$$\delta\bar{X}_\mu\ :\ \epsilon^{\nu\mu\alpha\beta}\partial_\nu a_{\alpha\beta} = 0. \tag{27}$$

Eq. (27) establishes that the two form $a$ is closed, i.e. $da = 0$. The Hodge-De Rham theorem says that the most general solution to this is obtained by requiring

$$a = dA + ls, \tag{28}$$

where $A$ is a one form, $l$ is a constant and $s$ is an harmonic two-form. Since in our case we naturally have one such form at our disposal, arising precisely from the chosen vacuum, we take

$$s = \frac{1}{2}C_{\mu\nu}dx^\mu dx^\nu, \tag{29}$$

which is certainly harmonic because it is constant. Calling $dA = f$ we have

$$a_{\alpha\beta}(x) = lC_{\alpha\beta} + f_{\alpha\beta}(x). \tag{30}$$

From (26) we obtain

$$\partial_\alpha \left( \frac{\partial \bar{V}}{\partial a_{\alpha\beta}} \right) = 0 \tag{31}$$

which are the generalization of Maxwell equations for the potential $A_\alpha$. From (26) we obtain $\partial_\nu \bar{X}_\mu - \partial_\mu \bar{X}_\nu$ yielding

$$\left( \partial_\rho \bar{X}_\sigma - \partial_\sigma \bar{X}_\rho \right) = \epsilon_{\rho\sigma\alpha\beta} \frac{\partial \bar{V}}{\partial a_{\alpha\beta}} \equiv R_{\rho\sigma}(A_\beta). \tag{32}$$

We verify that $R_{\rho\sigma}(A_\beta)$ can in fact be written as the left hand side of Eq. (32) by calculating $\epsilon^{\alpha\beta\rho\sigma}\partial_\beta R_{\rho\sigma}$. The result is zero in virtue of the equations of motion for $A_\beta$. Fixing the corresponding gauge freedom by setting $\partial^\mu \bar{X}_\mu = 0$, for example, we obtain

$$\partial^2 \bar{X}_\sigma = \partial_\rho R_{\rho\sigma}(A_\beta), \tag{33}$$

which completely determines the Lagrange multiplier $\bar{X}_\sigma$. Going back to the action (25), integrating by parts and substituting the constraint (27) we are left with the reduced effective action

$$S(A_\alpha) = -\int d^4x \, \left( \bar{V}(a_{\alpha\beta}) + V(C) \right), \qquad a_{\alpha\beta}(x) = lC_{\alpha\beta} + \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{34}$$

The resulting equations of motion are

$$\delta A_\alpha : \;\; \partial_\beta \left( \frac{\partial \bar{V}}{\partial a_{\beta\alpha}} \right) = 0, \tag{35}$$

which coincide with the previous ones in Eq. (31).

It is convenient to rename the dimensionless parameter $l = \xi - 1$. Substituting (30) in (34) and defining

$$C_{\mu\nu} = \frac{1}{2\xi} D_{\mu\nu}, \qquad \mathcal{B} = \frac{\beta}{4} > 0, \tag{36}$$

where

$$\beta = \frac{1}{2\left(\xi^2 - 1\right)\left(C^{\alpha\beta} C_{\alpha\beta}\right)}, \qquad \left(\xi^2 - 1\right)\left(C^{\alpha\beta} C_{\alpha\beta}\right) > 0, \tag{37}$$

we obtain our final effective action

$$S(A_\alpha) = \int d^4x \left( -\frac{1}{4} \left[ 1 - D^2 \mathcal{B} \right] D^2 - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \mathcal{B} \left[ \left( D_{\mu\nu} f^{\mu\nu} \right) + \left( f_{\mu\nu} f^{\mu\nu} \right) \right]^2 \right), \tag{38}$$

where $f_{\alpha\beta}(x) = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. We recognize the standard Maxwell kinetic term in the right hand side of Eq.(38). The only restriction now is $\mathcal{B} > 0$, with $D^2$ arbitrary.

Let us emphasize that the case of purely spontaneously broken Lorentz invariance really corresponds to the singular choice $\xi = 1, (l = 0)$. In this case, the corresponding action would not possess the standard kinetic term $-\frac{1}{4} f_{\mu\nu} f^{\mu\nu}$ but will start with the quadratic term $(D_{\mu\nu} f^{\mu\nu})^2$. This very interesting case is not considered here and its investigation is postponed for future work.

## IV. SYMMETRY ALGEBRAS ARISING FROM DIFFERENT CHOICES OF THE VACUUM

In this section we study the possible vacua allowed by the tensor symmetry breaking and we also identify the corresponding subgroups of the Lorentz group which are left invariant after the breaking. In order to do this, it is convenient to parameterize the background field $D_{\mu\nu}$ in terms of three dimensional components

$$D_{ij} = -\epsilon_{ijm} b_m, \qquad D_{0i} = e_i, \quad \epsilon_{123} = 1 \tag{39}$$

$$[D_{\mu\nu}] = \begin{bmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & -b_3 & b_2 \\ -e_2 & b_3 & 0 & -b_1 \\ -e_3 & -b_2 & b_1 & 0 \end{bmatrix} \tag{40}$$

which will mix when going to another reference frame via a passive Lorentz transformation. Since we have two vectors that determine a plane we choose a coordinate frame where

$$\mathbf{b} = (0,0,b), \qquad \mathbf{e}{=}(0, \quad e_2 = |\mathbf{e}|\sin\psi, \quad e_3 = |\mathbf{e}|\cos\psi), \tag{41}$$

That is to say, we have chosen the plane of the two vectors as the $(z-y)$ plane, with the vector $\mathbf{b}$ defining the $z$-direction and $\psi$ being the angle between $\mathbf{b}$ and $\mathbf{e}$. In this way the matrix representing the vacuum is

$$[D_{\mu\nu}] = \begin{bmatrix} 0 & 0 & e_2 & e_3 \\ 0 & 0 & -b & 0 \\ -e_2 & b & 0 & 0 \\ -e_3 & 0 & 0 & 0 \end{bmatrix}. \tag{42}$$

The most general infinitesimal generator $G$, including Lorentz transformations plus dilatations is

$$G = [G^\mu{}_\nu] = -i \begin{bmatrix} z & x_1 & x_2 & x_3 \\ x_1 & z & -y_3 & y_2 \\ x_2 & y_3 & z & -y_1 \\ x_3 & -y_2 & y_1 & z \end{bmatrix}. \tag{43}$$

Motivated by the work in Ref. [27] we are including conformal dilatations $D$ among our generators. Within this restricted Poincare algebra, this generator commutes with the remaining ones corresponding to pure Lorentz transformations and can be realized as a multiple of the identity. We do this in order to explore the possibility of having the largest possible invariant sub-algebra after the symmetry breaking.

The condition for the vacuum to be invariant under the infinitesimal transformations is

$$0 = G^\mu{}_\alpha D^{\alpha\nu} + G^\nu_{\ \alpha}\, D^{\mu\alpha} \tag{44}$$

and the resulting equations are:

$$x_2 b - y_2 e_3 = -y_3 e_2, \quad x_2 e_3 + y_2 b = x_3 e_2, \tag{45}$$
$$x_1 e_2 + 2zb = 0, \quad y_1 e_2 + 2z e_3 = 0, \tag{46}$$
$$-x_1 b + y_1 e_3 = 2z e_2, \quad x_1 e_3 + y_1 b = 0. \tag{47}$$

Next we consider the solutions of the above system of equations corresponding to the seven non-trivial possibilities dictated by the choices in the arrangement $(b,\ e_2,\ e_3)$ in the above coordinate system. In the sequel we use the notation of Ref.[28] for the Lorentz group generators.

### A. Case $b = e_3 = 0, e_2 \neq 0$

This leads to

$$z = 0, \quad x_1 = y_1 = 0, \qquad x_3 = y_3 = 0. \tag{48}$$

We have two free parameters, $x_2, y_2$ and the generator is

$$G = x_2\, K^2 - y_2\, J^2. \tag{49}$$

### B. Case $b = 0,\ e_3 \neq 0,\ \ e_2 = 0$

Here we have

$$z = 0, \quad x_1 = y_1 = 0, \quad x_2 = y_2 = 0, \tag{50}$$

so that we are left with a two parameter $(x_3,\ y_3)$ subalgebra with generator

$$G = x_3 K^3 - y_3 J^3. \tag{51}$$

### C. Case $b \neq 0, e_3 = 0, \ \ e_2 = 0$

This case is analogous to the previous one. We have

$$z = 0, \quad x_1 = y_1 = 0, \ x_2 = y_2 = 0, \tag{52}$$

with a two parameter $(x_3, \ y_3)$ subalgebra.

### D. Case $b = 0, e_3 \neq 0, \ e_2 \neq 0$

In this case we have

$$z(e_2^2 + e_2^2) \ = 0 \quad \to z = 0, \tag{53}$$

$$x_1 = 0, \ \ y_1 = 0, \ \ y_2 = y_3 \frac{e_2}{e_3}, \ \ x_2 = x_3 \frac{e_2}{e_3}, \tag{54}$$

yielding a two parameter $(x_3, y_3)$ subalgebra with generator

$$G = x_3 \left( \frac{e_2}{e_3} K^2 + K^3 \right) - y_3 \left( J^3 + \frac{e_2}{e_3} J^2 \right). \tag{55}$$

Calling

$$G_{x_3} = \frac{e_2}{e_3} K^2 + K^3, \ \ G_{y_3} = - \ \left( J^3 + \frac{e_2}{e_3} J^2 \right), \tag{56}$$

we can show that

$$[G_{x_3}, \ G_{y_3}] = 0. \tag{57}$$

### E. Case $b \neq 0, \ e_2 \neq 0, \ e_3 = 0$

Here we have

$$y_3 = -x_2 \frac{b}{e_2}, \quad x_3 = +y_2 \frac{b}{e_2} \ , \ y_1 = 0 \tag{58}$$

and the consistency condition

$$z(b^2 - e_2^2) = 0, \tag{59}$$

which leads to two possibilities

#### 1. *Subcase* $z = 0$

Here we have

$$y_3 = -x_2 \frac{b}{e_2}, \quad x_3 = +y_2 \frac{b}{e_2} \ , \ y_1 = 0, \ x_1 = 0 \tag{60}$$

and we are left with a two-parameter subgroup where

$$G = x_2 \left( K^2 + \frac{b}{e_2} J^3 \right) + y_2 \left( \frac{b}{e_2} K^3 - J^2 \right). \tag{61}$$

Calling

$$G_{x_2} = \left( K^2 + \frac{b}{e_2} J^3 \right), \quad G_{y_2} = \left( \frac{b}{e_2} K^3 - J^2 \right), \tag{62}$$

we can verify that

$$[G_{x_2}, \ \ G_{y_2}] = 0. \tag{63}$$

### 2. Subcase $z \neq 0$

Here we have

$$b^2 - e_2^2 = 0 \rightarrow b = se_2, \quad s = \pm 1, \tag{64}$$

which corresponds to a plane wave VEV.

Summarizing we have

$$y_1 = 0, \qquad x_1 = -2sz, \quad x_2 = -sy_3, \; y_2 = sx_3, \tag{65}$$

Here we are left with a three parameter Lie algebra ($z, \; x_3, \; y_3$) and the generator is

$$G = -z\left(2sK^1 + iI\right) + x_3\left(K^3 - sJ^2\right) - y_3\left(sK^2 + J^3\right). \tag{66}$$

Defining

$$G_z = -\left(2sK^1 + iI\right), \quad G_x = K^3 - sJ^2, \;\; G_y = -\left(J^3 + sK^2\right), \tag{67}$$

we obtain the algebra

$$[G_z, G_x] = 2iG_x, \;\; [G_z, G_y] = 2iG_y, \quad [G_x, G_y] = 0, \tag{68}$$

which is isomorphic to HOM(2).

The condition (64) implies that $D^2 = D_{\mu\nu}D^{\mu\nu} = 0$. Since $\mathcal{B}$ defined in Eq. (36) has to be finite and positive we have to be careful in the limiting procedure leading to this quantity. Also we need to have $C_{\mu\nu}C^{\mu\nu} \neq 0$, in order that the original parameters $\alpha$ and $\beta$ are well defined. Let us consider

$$e_2 = sb + \epsilon, \;\; e_3 = 0, \;\; \epsilon \rightarrow 0 \;, \tag{69}$$

such that

$$D^2 = 2(b^2 - e_2^2) = -4sb\epsilon. \tag{70}$$

The choice

$$\xi = \gamma\sqrt{\epsilon}, \tag{71}$$

with $\gamma$ arbitrary, guarantees that

$$\mathcal{B} = \frac{\gamma^2}{8sb} \tag{72}$$

is finite. Besides we must demand that

$$sb > 0. \tag{73}$$

Next we consider

$$C_{\mu\nu} = \frac{1}{2\gamma\sqrt{\epsilon}} D_{\mu\nu}, \quad C^2 = -\frac{4sb\epsilon}{4\gamma^2\epsilon} = -\frac{sb}{\gamma^2} < 0, \tag{74}$$

which ensures that $C^2$ is finite. The fact that $C^2$ is negative is consistent with the second condition in (37). Summarizing, in this case we have

$$\beta = \frac{\gamma^2}{2sb} > 0, \quad \alpha = \frac{1}{2}, \tag{75}$$

showing that the limiting procedure is well defined.

## F. Case $b \neq 0, \quad e_3 \neq 0, \; e_2 = 0$

We have

$$z = 0, \qquad x_1 = y_1 = 0, \qquad x_2 = y_2 = 0. \tag{76}$$

The free parameters here are $x_3, y_3$.

### G. Case $b \neq 0,\ e_2 \neq 0,\ e_3 \neq 0$

Here

$$y_3 = -x_2 \frac{b}{e_2} + y_2 \frac{e_3}{e_2}, \quad x_3 = x_2 \frac{e_3}{e_2} + y_2 \frac{b}{e_2}\ , \tag{77}$$

together with the consistency condition

$$z(b^2 - e_3^2 - e_2^2) = 0, \quad zbe_3 = 0, \tag{78}$$

which leads to $z = 0$. We have two free parameters: $x_2, y_2$ . The generators are

$$G = x_2 \left( K^2 + \frac{e_3}{e_2} K^3 + \frac{b}{e_2} J^3 \right) - y_2 \left( J^2 - \frac{b}{e_2} K^3 + \frac{e_3}{e_2} J^3 \right). \tag{79}$$

Defining

$$G'_{x_2} = \left( K^2 + \frac{e_3}{e_2} K^3 + \frac{b}{e_2} J^3 \right), \quad G'_{y_2} = -\left( J^2 - \frac{b}{e_2} K^3 + \frac{e_3}{e_2} J^3 \right), \tag{80}$$

we can verify that

$$\left[ G'_{x_2},\ G'_{y_2} \right] = 0. \tag{81}$$

Summarizing, all the two-parameter subalgebras that leave the vacuum invariant are isomorphic to $T(2)$, while the only three-parameter subalgebra, corresponding to the case (E-2), is isomorphic to $HOM(2)$**.**

### H. THE SYMMETRIES OF THE ACTION

To conclude this Section, let us consider the full effective action

$$L = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \mathcal{B} \left[ (D^{\mu\nu} f_{\mu\nu}) + (f_{\mu\nu} f^{\mu\nu}) \right]^2 \tag{82}$$

and verify the invariance under the Lorentz plus scale transformations that leave the vacuum invariant given by the generator (43) together with the condition (44). The term $f_{\mu\nu} f^{\mu\nu}$ is invariant under the full Lorentz group plus scale transformations, which includes also the second term in the parenthesis proportional to $\mathcal{B}$. Let us verify the invariance of the term $(D^{\mu\nu} f_{\mu\nu})$ under the transformation (44). We have

$$\delta \left( D^{\mu\nu} f_{\mu\nu} \right) = \left( \delta D^{\mu\nu} \right) f_{\mu\nu} + \left( D^{\mu\nu} \delta f_{\mu\nu} \right) = \left( D^{\mu\nu} \delta f_{\mu\nu} \right), \tag{83}$$

because (44) is the subset of transformations that leave the vacuum $D^{\mu\nu}$ invariant. The remaining transformations are given by those of a contravariant tensor

$$\left( D^{\mu\nu} \delta f_{\mu\nu} \right) = -D^{\mu\nu} \left( f_{\alpha\nu} G^{\alpha}{}_{\mu} + f_{\mu\alpha} G^{\alpha}{}_{\nu} \right) = -f_{\rho\sigma} \left( G^{\rho}{}_{\alpha} D^{\alpha\sigma} + G^{\sigma}{}_{\alpha} D^{\rho\alpha} \right) = 0 \tag{84}$$

because the last equation in the right hand side just reproduces the vacuum invariance condition (44). In these way, the symmetries of (82) are in fact those of the vacuum $D^{\mu\nu}$.

## V. THE EQUATIONS OF MOTION: THE PROPAGATING SECTOR

To study the propagation properties we consider only the quadratic terms in the effective Lagrangian

$$L_0 = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \mathcal{B} \left( f_{\mu\nu} D^{\mu\nu} \right)^2, \tag{85}$$

where we recall that $f_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. The equations are

$$\left( \partial^2 A_\beta - \partial_\beta \partial^\alpha A_\alpha \right) = -8\mathcal{B} D_{\alpha\beta} \partial^\alpha \left( D^{\mu\nu} \partial_\mu A_\nu \right). \tag{86}$$

We have verified the consistency of the above when taking $\partial^\beta$.

It is convenient to define

$$X = D^{\mu\nu}\partial_\mu A_\nu \tag{87}$$

and to introduce the notation

$$\begin{aligned} D_{\alpha k}\partial^\alpha &= D_k = \left[D_{0k}\partial_0 - D_{lk}\partial_l\right], && (88)\\ D_0 &= D_{i0}\partial_i. && (89)\end{aligned}$$

In this way we have

$$X = -\left(D_j A_j + D_0 A_0\right), \tag{90}$$

together with

$$D_0\partial_0 = -\partial_l D_l. \tag{91}$$

### A. Covariant formulation

The modified dispersion relations are found very easily by manipulating Eqs.(86). In momentum space

$$A_\mu(x) = \int d^4x\ A_\mu(k)e^{-ik_\mu x^\mu} \tag{92}$$

and choosing the Lorentz gauge, these equations reduce to

$$\begin{aligned} k^2 A_\mu + 2p_\mu(p^\nu A_\nu) = 0 \quad &, \quad p^\alpha \equiv 2\sqrt{\mathcal{B}}D^{\alpha\beta}k_\beta, && (93)\\ k^\nu A_\nu = 0 \quad &. && (94)\end{aligned}$$

The vector $p^\alpha$ is proportional to the momentum space version of the vector $D_\mu$ introduced in Eqs. (88) and (89).

Multiplying the first relation in (93) by $p^\mu$, it follows that

$$(k^2 + 2p^2)(p^\nu A_\nu) = 0. \tag{95}$$

Moreover $p^\nu A_\nu$ is gauge invariant. In fact, in coordinate space is proportional to $D^{\alpha\beta}f_{\alpha\beta}$ . So this component is physical and has a dispersion relation given by

$$k^2 + 2p^2 = 0. \tag{96}$$

If $(k^2 + 2p^2)$ is not zero in Eq.(95), it follows that $p^\nu A_\nu = 0$ . In four dimensions, this condition plus the Lorentz gauge leaves two degrees of freedom. But the Lorentz gauge permits a further gauge transformation with parameter $\lambda$ such that

$$\partial^2\lambda = 0, \tag{97}$$

which leaves only one degree of freedom as it should be. Moreover from the first equation in (93), this remaining degree of freedom satisfies

$$k^2 A_\mu = 0, \tag{98}$$

so its dispersion relation is

$$k^2 = 0. \tag{99}$$

The general solution of (95) is

$$p^\nu A_\nu = -\lambda_1(k)\delta(k^2 + 2p^2). \tag{100}$$

Putting it back into (93), we get

$$A_\mu = a_\mu(k)\delta(k^2) + 2\frac{\lambda_1(k)}{k^2}\delta(k^2 + 2p^2)p_\mu, \qquad a_\mu(k)k^\mu = 0, a_\mu(k)p^\mu = 0. \tag{101}$$

From (101), we obtain the electromagnetic tensor

$$f_{\mu\nu} = (k_\mu a_\nu(k) - k_\nu a_\mu(k))\delta(k^2) + 2\frac{\lambda_1(k)}{k^2}\delta(k^2 + 2p^2)(k_\mu p_\nu - k_\nu p_\mu). \tag{102}$$

It represents a plane wave with dispersion relation $k^2 = 0$ (the magnetic and electric fields are the standard ones, perpendicular to $\mathbf{k}$), plus a plane wave propagating in the direction $\mathbf{k}$ with dispersion relation $k^2 + 2p^2 = 0$.

The fields for the second wave are

$$E_j = f_{0j} = 2(k_0 p_j - k_j p_0)\frac{\lambda_1(k)}{k^2}, \qquad B_j = 2\epsilon_{jkl}(k_k p_l)\frac{\lambda_1(k)}{k^2}. \tag{103}$$

Notice that $\mathbf{E}$ is perpendicular to $\mathbf{B}$, $\mathbf{B}$ is perpendicular to $\mathbf{k}$, but $\mathbf{E}$ is not necessarily orthogonal to $\mathbf{k}$. Moreover, this wave exists only if $k^2 \neq 0$.

### B. 3+1 Decomposition

An alternative way to determine the polarization of the plane wave solutions is by splitting Eq. (86) into the components $0, i$, which yields

$$\left(-\nabla^2 + 2bD_0^2\right) A_0 - \partial_0 \partial^i A_i = -8\mathcal{B} D_0 D_j A_j, \tag{104}$$

$$\left(\partial^2 A_k - \partial_k \partial_0 A_0 + \partial_k \partial_i A_i\right) = 8\mathcal{B} D_k \left(D_j A_j + D_0 A_0\right). \tag{105}$$

We choose the Coulomb gauge

$$\partial_r A_r = 0. \tag{106}$$

The final equations are

$$\left(-\nabla^2 + 8\mathcal{B}D_0^2\right) A_0 = -8\mathcal{B} D_0 D_j A_j, \tag{107}$$

$$\left(\partial^2 A_k - \partial_k \partial_0 A_0\right) = 8\mathcal{B} D_k \left(D_j A_j + D_0 A_0\right). \tag{108}$$

As in the usual electrodynamics $A_0$ can be expressed as an instantaneous function of $A_j$

$$A_0 = -8\mathcal{B}\frac{1}{(-\nabla^2 + 8\mathcal{B}D_0^2)} D_0 D_j A_j. \tag{109}$$

The final vector equation leads to

$$\partial^2 A_k = \left(-\nabla^2 \delta_{kl} + \partial_k \partial_l\right) \frac{8\mathcal{B}}{(-\nabla^2 + 8\mathcal{B}D_0^2)} D_l D_j A_j, \tag{110}$$

where it is a direct matter to verify the Coulomb gauge condition.

The electromagnetic fields are given by

$$\begin{aligned}
\mathbf{A} &= \{A^i\}, &(111)\\
\mathbf{E} &= \{E_i\}, \ \ E_i = f_{0i} = \partial_0 A_i - \partial_i A_0, \ \ \mathbf{E} = -\partial_0 \mathbf{A} - \nabla A^0, &(112)\\
\mathbf{B} &= \{B_i\}, \ \ B_k = -\frac{1}{2}\epsilon_{klm} f_{lm} = \epsilon_{klm}\partial_l A^m, \ \ \epsilon_{123} = +1, \ \ \mathbf{B} = \nabla \times \mathbf{A}, &(113)
\end{aligned}$$

which reduces to

$$E_r = -i\omega A_r - ik_r A_0, \quad B_k = -i\epsilon_{klm} k_l A_m \tag{114}$$

in momentum space with the notation $\mathbf{k} = \{k_i\}$ in the three dimensional subspace.

### C. Modified Maxwell equations

They are

$$\nabla\cdot\mathbf{E}+8\mathcal{B}\left(\mathbf{e}\cdot\nabla\right)\left(\,\mathbf{B}\cdot\mathbf{b}-\mathbf{E}\cdot\mathbf{e}\right)=4\pi\rho \quad , \tag{115}$$

$$\nabla\times(\mathbf{B}+8\mathcal{B}\mathbf{b}(\,\mathbf{B}\cdot\mathbf{b}-\mathbf{E}\cdot\mathbf{e}))-\frac{1}{c}\frac{\partial}{\partial t}[\mathbf{E}+8\mathcal{B}\mathbf{e}(\,\mathbf{B}\cdot\mathbf{b}-\mathbf{E}\cdot\mathbf{e})]=\frac{4\pi}{c}\mathbf{J} \quad , \tag{116}$$

$$\nabla\cdot\mathbf{B}=0 \quad , \tag{117}$$

$$\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0 \quad . \tag{118}$$

In the notation of electrodynamics in a medium we can introduce

$$\mathbf{D}=\mathbf{E}+8\mathcal{B}\mathbf{e}(\,\mathbf{B}\cdot\mathbf{b}-\mathbf{E}\cdot\mathbf{e}), \quad \mathbf{H}=\mathbf{B}+8\mathcal{B}\mathbf{b}(\mathbf{B}\cdot\mathbf{b}-\mathbf{E}\cdot\mathbf{e}), \tag{119}$$

such that

$$\begin{aligned}\nabla\cdot\mathbf{D} &= 4\pi\rho, \quad \nabla\times\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=\frac{4\pi}{c}\mathbf{J}, & (120)\\ \nabla\cdot\mathbf{B} &= 0, \quad \nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0. & (121)\end{aligned}$$

In components

$$\begin{aligned}D_i &= (\delta_{ij}-8\mathcal{B}e_ie_j)E_j+8\mathcal{B}e_ib_jB_j, & (122)\\ H_i &= (\delta_{ij}+8\mathcal{B}b_ib_j)B_j-8\mathcal{B}b_ie_jE_j. & (123)\end{aligned}$$

Particular cases are

$$\begin{aligned}b_j &= 0\rightarrow D_i=(\delta_{ij}-8\mathcal{B}e_ie_j)E_j,\ H_i=B_i\ , & (124)\\ e_i &= 0\rightarrow \quad D_i=E_i, \quad H_i=(\delta_{ij}+8\mathcal{B}b_ib_j)B_j. & (125)\end{aligned}$$

The first case induces a modification in Coulomb law

$$\mathbf{B}=\mathbf{0}, \quad \nabla\cdot(\mathbf{E}-8\mathcal{B}\mathbf{e}\left(\mathbf{e}\cdot\mathbf{E}\right))=4\pi\rho, \ \ \mathbf{E}=-\nabla\Phi, \tag{126}$$

$$\left(\nabla^2-8\mathcal{B}\left(e_i\partial_i\right)^2\right)\Phi=-4\pi\rho. \tag{127}$$

## VI. SUMMARY OF THE DISPERSION RELATIONS AND ELECTROMAGNETIC FIELDS

Here we use the previous decomposition of the background field $D_{\mu\nu}$ in terms of three dimensional components together with the $3+1$ formulation described in subsection (IV-B), to search for plane wave solutions of the Maxwell equations. We assume the space-time dependence of any field to be proportional to $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, where $\mathbf{k}$ is the momentum of the wave and we choose the potential $\mathbf{A}$ in the Coulomb gauge. The notation is

$$\mathbf{b}=\{b_m\}, \ \ \mathbf{e}=\{e_m\}\ . \tag{128}$$

The expressions of Eqs. (88) , (89) in momentum space are

$$\begin{aligned}\{D_k\} &= \mathbf{D}=-i\left[\omega\mathbf{e}-\mathbf{k}\times\mathbf{b}\right], & (129)\\ D_0 &= -i\mathbf{e}\cdot\mathbf{k}. & (130)\end{aligned}$$

There are two main cases

(i) $D_jA_j=0$: In this case the dispersion relation is $\omega=|\mathbf{k}|$. The fields are

$$\mathbf{B}=\gamma\{(\mathbf{e}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}-\mathbf{e}+\mathbf{b}\times\hat{\mathbf{k}}\} \quad , \tag{131}$$

$$\mathbf{E}=\gamma\{(\mathbf{b}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}-\mathbf{b}-\mathbf{e}\times\hat{\mathbf{k}}\} \quad . \tag{132}$$

(ii) $D_j A_j \neq 0$ : In this case the dispersion relation and the fields are

$$\omega = |\mathbf{k}| \frac{\sqrt{256\mathcal{B}^2(\hat{\mathbf{k}}\cdot(\mathbf{b}\times\mathbf{e}))^2 - 4(1-8\mathcal{B}\mathbf{e}^2)\{8\mathcal{B}[(\hat{\mathbf{k}}\cdot\mathbf{e})^2 - (\hat{\mathbf{k}}\times\mathbf{b})^2] - 1\}} - 16\mathcal{B}\hat{\mathbf{k}}\cdot(\mathbf{b}\times\mathbf{e})}{2(1-8\mathcal{B}\mathbf{e}^2)}, \tag{133}$$

$$\mathbf{B} = \gamma\{\mathbf{k}\times(\mathbf{k}\times\mathbf{b}) - \omega\mathbf{k}\times\mathbf{e}\} \quad , \tag{134}$$

$$\mathbf{E} = \gamma\{\omega\mathbf{k}\times\mathbf{b} - \omega^2\mathbf{e} + \mathbf{k}(\mathbf{e}\cdot\mathbf{k})\} \quad . \tag{135}$$

In both cases $\gamma$ is an arbitrary constant. For small $\mathcal{B}$, the dispersion relation (133) reduces to

$$\omega = |\mathbf{k}| \left[1 + \mathcal{B}\left(8\hat{\mathbf{k}}\cdot(\mathbf{e}\times\mathbf{b}) + 4(\hat{\mathbf{k}}\times\mathbf{b})^2 + 4(\hat{\mathbf{k}}\times\mathbf{e})^2\right)\right] . \tag{136}$$

The above expression clearly shows that the model is stable in the small Lorentz invariance violation approximation, where the quantities $\mathcal{B}e^2, \mathcal{B}b^2, \mathcal{B}|\mathbf{e}||\mathbf{b}|$ are very small compared to one.

The anisotropic velocity of light arising from the above dispersion relation is

$$\begin{aligned} \nabla_{\mathbf{k}}\omega \ &= \ \mathbf{c}(\hat{\mathbf{k}}) = \hat{\mathbf{k}}\left(1 + 8\mathcal{B}\left(b^2+e^2\right) - 4\mathcal{B}\left((\hat{\mathbf{k}}\times\mathbf{b})^2 + (\hat{\mathbf{k}}\times\mathbf{e})^2\right)\right) \\ &+8\mathcal{B}\left(\mathbf{e}\times\mathbf{b}\right) \ -8\mathcal{B}\left(\mathbf{b}\cdot\hat{\mathbf{k}}\right)\mathbf{b} - 8\mathcal{B}\left(\mathbf{e}\cdot\hat{\mathbf{k}}\right)\mathbf{e}. \end{aligned} \tag{137}$$

Next we consider the separate cases where $\mathbf{e}\neq 0,\ \mathbf{b}=0,\ \ \mathbf{e}=0,\ \mathbf{b}\neq 0,$ and $\mathbf{e}\neq 0,\ \mathbf{b}\neq 0.$ The results are:

## A. Case I: $\mathbf{b}=0,\ \ \mathbf{e}\neq 0$

### 1. *CASE (I-1)* $\mathbf{e}\cdot\mathbf{A}\neq 0,\ \ \mathbf{D}\cdot\mathbf{A}\ \neq 0$

We find the dispersion relation

$$\omega_1^2 = \frac{1-8\mathcal{B}\mathbf{e}^2\cos^2\theta}{1-8\mathcal{B}\mathbf{e}^2}\mathbf{k}^2, \tag{138}$$

which agrees with the one obtained from (95). Here $\theta$ is the angle between $\mathbf{k}$ and $\mathbf{e}$. Assuming the corrections terms to be very small we can rewrite (138) as

$$\omega_1 = |\mathbf{k}|\left(1 + 4\mathcal{B}\mathbf{e}^2\sin^2\theta\right). \tag{139}$$

The electromagnetic fields in momentum space are

$$\mathbf{E} = \alpha\left[\mathbf{e} - \frac{(\mathbf{e}\cdot\mathbf{k})}{\omega_1^2}\mathbf{k}\right], \qquad \mathbf{B} = \frac{\alpha}{\omega_1}\left[\mathbf{k}\times\mathbf{e}\right], \tag{140}$$

where $\alpha$ is a dimensionless constant. The following orthogonality relations are satisfied

$$\mathbf{E}\cdot\mathbf{B}=0, \quad \hat{\mathbf{k}}\cdot\mathbf{E}\neq 0, \quad \hat{\mathbf{k}}\cdot\mathbf{B}=0, \tag{141}$$

together with the properties

$$\mathbf{E}\times\mathbf{B}\sim\ [\mathbf{k} - 8\mathcal{B}(\mathbf{e}\cdot\mathbf{k})\,\mathbf{e}] \equiv \mathbf{m}, \tag{142}$$

$$\mathbf{m}\cdot\mathbf{B}=0, \ \ \mathbf{m}\cdot\mathbf{E}=0. \tag{143}$$

The ratio between the amplitudes is

$$\frac{|\mathbf{E}|^2}{|\mathbf{B}|^2} = \frac{\sin^2\theta + \left(1-8\mathcal{B}\mathbf{e}^2\right)^2\cos^2\theta}{(1-8\mathcal{B}\mathbf{e}^2\cos^2\theta)\,(1-8\mathcal{B}\mathbf{e}^2)}. \tag{144}$$

### 2. CASE (I-2) $\mathbf{e}\cdot\mathbf{A}=0,\ \ \mathbf{D}\cdot\mathbf{A}=0$

The corresponding dispersion relation is

$$\omega_2^2=\mathbf{k}^2, \tag{145}$$

in accordance with (99), with the electromagnetic fields

$$\mathbf{E}=\beta\left(\hat{\mathbf{k}}\times\mathbf{e}\right),\quad \mathbf{B}=\beta\,\hat{\mathbf{k}}\times\left(\hat{\mathbf{k}}\times\mathbf{e}\right), \tag{146}$$

where $\beta$ is an arbitrary dimensionless constant. Also we have

$$|\mathbf{E}|=|\mathbf{B}|=\beta|\mathbf{e}|\sin\theta. \tag{147}$$

The orthogonality relations are

$$\hat{\mathbf{k}}\cdot\mathbf{E}=0,\quad \hat{\mathbf{k}}\cdot\mathbf{B}=0,\quad \mathbf{E}\cdot\mathbf{B}=\mathbf{0}, \tag{148}$$

similarly to the standard case. Also

$$\frac{|\mathbf{E}|^2}{|\mathbf{B}|^2}=1. \tag{149}$$

## B. CASE II: $\mathbf{e}=0,\ \ \mathbf{b}\neq 0,\ \ D_0=0,$

### 1. CASE (II-1) $\mathbf{b}\cdot\mathbf{A}\neq 0\implies \mathbf{D}\cdot\mathbf{A}=0$

We find the dispersion relation

$$\omega_3^2=\mathbf{k}^2. \tag{150}$$

and we have

$$\mathbf{E}=\gamma\left[\hat{\mathbf{k}}\times\left(\hat{\mathbf{k}}\times\mathbf{b}\right)\right],\quad \mathbf{B}=-\gamma\left(\hat{\mathbf{k}}\times\mathbf{b},\right) \tag{151}$$

with $\gamma$ an arbitrary dimensionless constant. Also

$$\frac{|\mathbf{E}|^2}{|\mathbf{B}|^2}=1,\quad \hat{\mathbf{k}}\cdot\mathbf{E}=\hat{\mathbf{k}}\cdot\mathbf{B}=0,\quad \mathbf{E}\cdot\mathbf{B}=0. \tag{152}$$

### 2. CASE (II-2) $\mathbf{b}\cdot\mathbf{A}=0\implies\mathbf{D}\cdot\mathbf{A}\neq\mathbf{0}$

Here the dispersion relation is

$$\omega_4^2=\mathbf{k}^2\left(1+8\mathcal{B}\mathbf{b}^2\sin^2\theta\right), \tag{153}$$

with $\theta$ being the angle between $\mathbf{k}$ and $\mathbf{b}$. The above result agrees with that obtained from (95). The electromagnetic fields are determined, up to the arbitrary constant $\delta$, by

$$\mathbf{E}=\frac{\delta}{\omega_4}\left(\mathbf{k}\times\mathbf{b}\right), \tag{154}$$

$$\mathbf{B}=\frac{\delta}{\omega_4^2}\,\mathbf{k}\times\left(\mathbf{k}\times\mathbf{b}\right) \tag{155}$$

and they satisfy

$$\hat{\mathbf{k}}\cdot\mathbf{E}=0,\quad \hat{\mathbf{k}}\cdot\mathbf{B}=0,\quad \mathbf{E}\cdot\mathbf{B}=0, \tag{156}$$

$$\frac{|\mathbf{E}|^2}{|\mathbf{B}|^2}=\frac{\omega_4^2}{|\mathbf{k}|^2}=\left(1+8\mathcal{B}\mathbf{b}^2\sin^2\theta\right). \tag{157}$$

Notice that half of the cases previously considered violate rotational invariance, with the exception of those having unmodified dispersion relations together with standard properties of the propagating electromagnetic fields.

### C. CASE III: $\mathbf{b}^2=\mathbf{e}^2$, $\mathbf{e}\cdot\mathbf{b}=\mathbf{0}$

We consider this case corresponding to a plane wave vacuum because the final theory remains invariant under the three-parameter subgroup HOM(2).

We choose a reference frame such that

$$\hat{\mathbf{n}}{=}(1,0,0),\quad \mathbf{b}{=}(0,0,b),\quad \mathbf{e}{=}(0,sb,0),\ s=\pm 1, \tag{158}$$

$$\mathbf{k}\cdot\mathbf{A}=\mathbf{0} \tag{159}$$

$$D_jA_j=ib\left(\left(k_1-\omega s\right)A_2-k_2A_1\right) \tag{160}$$

#### 1. Case $D_kA_k=0$.

The dispersion relations are the usual ones. From the master formulae (116-117), the fields are

$$\mathbf{B}=\gamma\{(s(\mathbf{b}\times\hat{\mathbf{n}})\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}-s(\mathbf{b}\times\hat{\mathbf{n}})+\mathbf{b}\times\hat{\mathbf{k}}\}\quad, \tag{161}$$

$$\mathbf{E}=\gamma\{(\mathbf{b}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}-\mathbf{b}+s\hat{\mathbf{k}}\times(\mathbf{b}\times\hat{\mathbf{n}})\}\quad. \tag{162}$$

#### 2. Case $D_kA_k\neq 0$.

The exact modified dispersion relations are

$$\omega_5=\frac{1}{\left(1-8\mathcal{B}b^2\right)}\left(\sqrt{\left(\mathbf{k}^2-8\mathcal{B}\mathbf{b}^2\left(\mathbf{k}^2-k_1^2\right)\right)}+8\mathcal{B}b^2sk_1\right). \tag{163}$$

In the small violation approximation Eq. (163) reduces to

$$\omega_5=|\mathbf{k}|\left(1+4\mathcal{B}b^2\left[1+s\frac{k_1}{|\mathbf{k}|}\right]^2\right). \tag{164}$$

From the formulae (118-119), the fields are

$$\mathbf{B}=\gamma\{\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{b})-\omega_5 s\hat{\mathbf{k}}\times(\mathbf{b}\times\hat{\mathbf{n}})\}\quad, \tag{165}$$

$$\mathbf{E}=\gamma\{\omega_5\left(\hat{\mathbf{k}}\times\mathbf{b}\right)-\omega_5^2 s(\mathbf{b}\times\hat{\mathbf{n}})+s\hat{\mathbf{k}}((\mathbf{b}\times\hat{\mathbf{n}})\cdot\hat{\mathbf{k}})\}\quad. \tag{166}$$

## VII. THE MODEL AS A SECTOR OF THE SME

In order to impose some bounds upon the parameters of the model it is convenient to recast the quadratic sector of the action (38) in the language of the Standard Model Extension and to make use of the numerous experimental constraints derived from it ( see for example Ref. [24]) and summarized in Table II: Photon-sector summary, of Ref. [29]. To begin with, let us recall the dimension of the fields and parameters involved in the model

$$[A]=m,\qquad [e]=[b]=m^2,\qquad [\mathcal{B}]=\frac{1}{m^4},\qquad \left[\mathcal{B}e^2\right]=0. \tag{167}$$

Next we make the identification

$$-\mathcal{B}\left(f_{\mu\nu}D^{\mu\nu}\right)^2=-\frac{1}{4}\left(k_F\right)^{\kappa\lambda\mu\nu}f_{\kappa\lambda}f_{\mu\nu}, \tag{168}$$

where the tensor $(k_F)^{\kappa\lambda\mu\nu}$, with 19 independent components, has all the symmetries of the Riemann tensor and a vanishing double trace. In terms of the matrix elements $D_{\mu\nu}$ characterizing the vacuum expectation values of the electromagnetic tensor we obtain

$$\left(k_F\right)^{\kappa\lambda\mu\nu}=4\mathcal{B}\left[D^{\kappa\lambda}D^{\mu\nu}+\frac{1}{2}\left(D^{\kappa\mu}D^{\lambda\nu}-D^{\lambda\mu}D^{\kappa\nu}\right)\right]-\frac{1}{2}\mathcal{B}D^2\left(\eta^{\kappa\mu}\eta^{\lambda\nu}-\eta^{\lambda\mu}\eta^{\kappa\nu}\right), \tag{169}$$

where

$$D^2 \equiv D_{\alpha\beta}D^{\alpha\beta} = 2(\mathbf{b}^2 - \mathbf{e}^2). \tag{170}$$

Next we have to identify the appropriate combinations of the components of $(k_F)^{\kappa\lambda\mu\nu}$ which are bounded. A first step in that direction constitutes the definitions of the components $(\kappa_{DE})^{jk}$, $(\kappa_{HB})^{jk}$ and $(\kappa_{DB})^{jk} = -(\kappa_{HE})^{kj}$, which are written in terms of our parameters $\mathbf{e}, \mathbf{b}$ and $\mathcal{B}$ as follows

$$(\kappa_{DE})^{jk} \equiv -2\,(k_F)^{0j0k} = -12\mathcal{B}e_je_k - \mathcal{B}D^2\delta_{jk}, \tag{171}$$

$$(\kappa_{HB})^{jk} \equiv \frac{1}{2}\epsilon_{jpq}\epsilon_{krs}\,(k_F)^{pqrs} = 12\mathcal{B}b_jb_k - \mathcal{B}D^2\delta_{jk}, \tag{172}$$

$$(\kappa_{DB})^{jk} = -(\kappa_{HE})^{kj} = \epsilon_{kpq}(k_F)^{0jpq} = 12\mathcal{B}(e_jb_k - \frac{1}{3}\left(\mathbf{e}\cdot\mathbf{b}\right)\delta_{jk}), \quad tr\left[\kappa_{DB}^{jk}\right] = 0. \tag{173}$$

The final combinations in terms of which the bounds are presented turn out to be

$$(\bar\kappa_{e+})^{jk} = \frac{1}{2}\left(\kappa_{DE} + \kappa_{HB}\right)^{jk} = 6\mathcal{B}\left[b_jb_k - e_je_k - \frac{1}{3}\left(\mathbf{b}^2 - \mathbf{e}^2\right)\delta_{jk}\right] < 10^{-32}, \tag{174}$$

$$\begin{aligned}(\bar\kappa_{e-})^{jk} &= \frac{1}{2}\left(\kappa_{DE} - \kappa_{HB}\right)^{jk} - \frac{1}{3}\delta_{jk}tr(\kappa_{DE}) \\ &= -6\mathcal{B}\left[(b_jb_k + e_je_k) - \frac{1}{3}\delta_{jk}(\mathbf{b}^2 + \mathbf{e}^2)\right] < 10^{-16}\,,\end{aligned} \tag{175}$$

$$(\bar\kappa_{o+})^{jk} = \frac{1}{2}\left(\kappa_{DB} + \kappa_{HE}\right)^{jk} = 6\mathcal{B}(e_jb_k - e_kb_j) < 10^{-12}, \tag{176}$$

$$(\bar\kappa_{o-})^{jk} = \frac{1}{2}\left(\kappa_{DB} - \kappa_{HE}\right)^{jk} = 6\mathcal{B}\left[(e_jb_k + e_kb_j) - \frac{2}{3}\left(\mathbf{e}\cdot\mathbf{b}\right)\delta_{jk}\right] < 10^{-32}, \tag{177}$$

$$\bar\kappa_{tr} = \frac{1}{3}tr\left(k_{DE}\right)^j = -2\mathcal{B}(\mathbf{e}^2 + \mathbf{b}^2) < 10^{-15}. \tag{178}$$

The bounds for $\bar\kappa_{tr}$ have recently been improved from $10^{-7}$ [29], to $10^{-11}$ [30], and finally to $10^{-15}$ [31]. In quoting our results arising from the bounds upon $(\bar\kappa_{e+})^{jk}$, $(\bar\kappa_{o-})^{jk}$ and $\bar\kappa_{tr}$ we have chosen our reference frame in such a way that only the light-cone gets modified by the LIV parameters, while the propagation of the fermioms is the standard one [24],[32].

All matrices from (174) to (177) are traceless, with $(\bar\kappa_{0+})^{jk}$ being antisymmetric (3 independent components) while the three remaining ones are symmetric (5 independent components each). The above combinations (174) to (178) constitute a convenient alternative way to display the 19 independent components of the tensor $(k_F)^{\alpha\beta\mu\nu}$. The bounds are obtained in terms of such combinations referred to the Standard Inertial Reference Frame defined in Ref. [24] and are written at the end of each of the above equations.

Notice that the bilinears which are odd under the duality transformations

$$\mathbf{e} \rightarrow \mathbf{b}, \quad \mathbf{b} \rightarrow -\mathbf{e}, \tag{179}$$

are much more suppressed than those which are even. In this way, even though our model is not duality invariant, this transformation seems to explain the above mentioned hierarchy in the LIV parameters exhibited in Eqs. (174)-(178). In order to obtain some specific consequences of the bounds (174) to (178) it is convenient to express them in the coordinate system defined by (41). Also we introduce the notation

$$\mathbf{x} = \sqrt{6\mathcal{B}}\,\mathbf{e}\times 10^{16}, \quad \mathbf{y} = \sqrt{6\mathcal{B}}\,\mathbf{b}\times 10^{16}, \qquad x = |\mathbf{x}|, \quad y = |\mathbf{y}| \tag{180}$$

We consider only absolute values of the related quantities and we focus upon the stringent constraints in Eqs. (174) and (177). The non-trivial contributions are

$$\left|(\tilde\kappa_{e+})^{11}\right| = \left|(x^2 - y^2)\right| < 3, \tag{181}$$

$$\left|(\tilde\kappa_{e+})^{22}\right| = \left|\left(1 - 3\sin^2\psi\right)x^2 - y^2\right| < 3, \tag{182}$$

$$\left|(\tilde\kappa_{e+})^{33}\right| = \left|2y^2 + \left(1 - 3\cos^2\psi\right)x^2\right| < 3, \tag{183}$$

$$\left|(\tilde\kappa_{e+})^{23}\right| = \left|x^2\sin 2\psi\right| < 2, \tag{184}$$

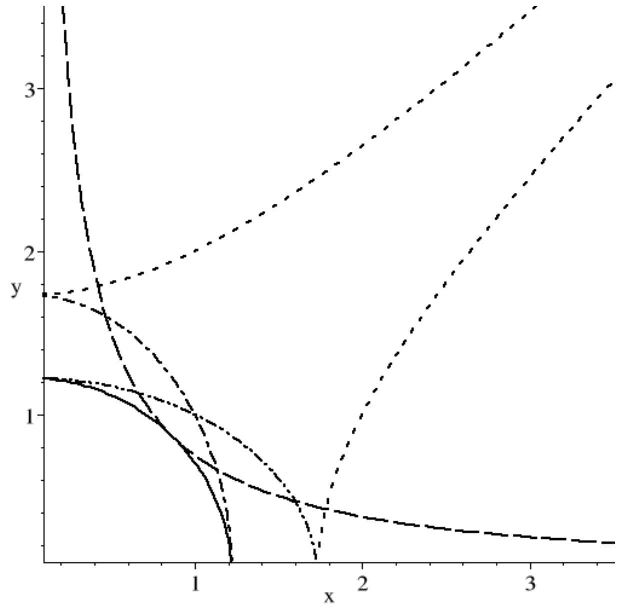


FIG. 1: Boundaries of the allowed region obtained from the constraints in the parameters $\tilde{\kappa}_{e+}^{jk}$ and $\tilde{\kappa}_{o-}^{jk}$. The allowed region is on the inside the dashed, dot-dashed and dot-dot-dashed lines. An excellent approximation for it is the sector of the circle shown in solid line [33].

together with

$$\begin{aligned}
(\tilde{\kappa}_{o-})^{11} &= (\tilde{\kappa}_{o-})^{22} = |xy\cos\psi| < \frac{3}{2}, && (185)\\
(\tilde{\kappa}_{o-})^{33} &= |xy\cos\psi| < \frac{3}{4}, && (186)\\
(\tilde{\kappa}_{o-})^{23} &= |xy\sin\psi| < 1. && (187)
\end{aligned}$$

The allowed region in the $(x-y)$ plane is shown in Fig.1, where we plot the boundaries of the corresponding inequalities. In expressions (184), (185)-(187) we consider the lower bound corresponding to the maximum value of the trigonometric function on the LHS of the inequality. This leads to

$$x < \sqrt{2}, \quad |xy| < \frac{3}{4} < 1 < \frac{3}{2}. \tag{188}$$

The boundary $xy = 3/4$ is plotted in dashed line. The most stringent bound from Eq. (183) results when $\psi = \pi/2$ and corresponds to

$$y < \sqrt{\frac{3-x^2}{2}}. \tag{189}$$

The boundary is shown in dot-dot-dashed line. The most stringent bound from (182) arises again from $\psi = \pi/2$ and corresponds to

$$y < \sqrt{3-2x^2}, \tag{190}$$

which boundary is plotted in dot-dashed line. The expression (181) does not provide additional bounds and is plotted for completeness. The corresponding boundaries are $y_\pm = \sqrt{x^2 \pm 3}$ shown in dotted lines in Fig. 1. An upper bound including all previous ones is given by the interior of the circle

$$y = \sqrt{\frac{3}{2} - x^2}, \tag{191}$$

shown in solid line in Fig. 1 and which translates into the restriction

$$\mathcal{B}(\mathbf{e}^2 + \mathbf{b}^2) = \frac{1}{M_{\mathcal{B}}^4}(\mathbf{e}^2 + \mathbf{b}^2) \ < 2.5 \times 10^{-33}. \tag{192}$$

This bound incorporates all the constraints established in Eqs.(174)-(178). All the above relations are valid in the Standard Inertial Reference Frame centered in the Sun [24].

## VIII. SUMMARY AND FINAL COMMENTS

We study a novel way of implementing a model with spontaneously broken Lorentz symmetry by introducing a constant vacuum expectation value (VEV) of the field strength $\langle F_{\mu\nu}\rangle = C_{\mu\nu}$ instead of imposing a VEV of the electromagnetic potential $\langle A_\mu\rangle$ as frequently investigated in the literature. In this way, our model preserves gauge invariance from the very beginning. We start from the effective Lagrangian (13) describing non-linear electrodynamics and containing a potential $V_{eff}(F_{\mu\nu})$, which is argued to arise after integrating massive gauge bosons and fermions in an underlying conventional theory. This approach is an extension of the model in Refs. [7, 8], which includes all quartic fermion interactions allowed by the Dirac algebra. For simplicity we have considered only one fermion species. The subsequent integration of them produces the effective gauge invariant potential which is bounded in the high-intensity field limit and has an stable minimum. In order to investigate the dynamical consequences of the electrodynamics constructed around such minimum we have chosen the standard Ginzburg-Landau parametrization for $V_{eff}(F_{\mu\nu})$. We have verified that the extremum conditions in the potential corresponds to the extremum conditions of the energy arising from (13), under the requirement that the involved fields are constant at the extreme points. Next we expand the fields around the minimum and arrive to the spontaneously broken action (25) from where we can further discuss the resulting non-linear electrodynamics, which differs from the standard one described in Ref. [26], for example. The action (25) contains the field strength $a_{\mu\nu}$ ($\mathbf{E}$ , $\mathbf{B}$), together with the auxiliary field $\bar{X}_\mu$ which equations of motion demand that the two-form $a$ is such that $da = 0$, thus introducing the electromagnetic potentials. The constitutive relations expressing the electromagnetic excitations $\mathbf{D}$ and $\mathbf{H}$ , are identified via the equations of motions (26), (31) and provide a direct connection with the field strengths $\mathbf{E}$ and $\mathbf{B}$. After eliminating the auxiliary field $\bar{X}_\mu$ and writing the most general solution to the condition $da = 0$, we arrive at the action (38), which is the starting point of our subsequent analysis. Some redefinitions allow the symmetry breaking parameters to be characterized by the constant antisymmetrical tensor $D_{\mu\nu}$ (proportional to the VEV $C_{\mu\nu}$), plus an additional constant $\mathcal{B}$.

The search for the subgroups under which the model remains invariant after the symmetry breaking includes the generator of dilatations plus the standard Lorentz generators. In this restricted subalgebra the former commutes with the remaining ones, which allows its realization as a multiple of the identity. There is only one case, given in the subsubsection IV.E-2, which incorporates this generator, resulting in the breaking of the Lorentz group to the three generators subgroup $HOM(2)$. All the other cases break to a subgroup isomorphic to $T(2)$, with two generators.

The modified photon dispersion relations together with the corresponding plane wave polarizations are classified according to the value of $D_{i\alpha}k^\alpha A^i$, where $k^\alpha$ is the plane wave four momentum and $A^i$ is the corresponding vector potential in the Coulomb gauge. The amplitudes of the propagating fields are written in terms of those parameterizing $D_{\mu\nu}$. The case $D_{i\alpha}k^\alpha A^i = 0$ leads to dispersion relations $\omega = |\mathbf{k}|$ with the triad $\mathbf{E}, \mathbf{B}, \ \mathbf{k}$ having standard orthogonality properties. The situation $D_{i\alpha}k^\alpha A^i \neq 0$ produces the following properties for the vectors involved: $\mathbf{E}$ is perpendicular to $\mathbf{B}$, $\mathbf{B}$ is perpendicular to $\mathbf{k}$, but $\mathbf{E}$ is not necessarily orthogonal to $\mathbf{k}$. The corresponding dispersion relations are of the form

$$\omega_\pm = |\mathbf{k}| \ F_\pm, \tag{193}$$

where $F_\pm$ is independent of $|\mathbf{k}|$, being only function of the angles between $\mathbf{k}$ and $\mathbf{e}$ , $\mathbf{k}$ and $\mathbf{b}$, $\mathbf{k}$ and $\mathbf{e}\times\mathbf{b}$. In this way our model predicts anisotropy in the speed of light. Also, as shown in Eq. (136) the model is stable in the small Lorentz invariance violation approximation where the quantities $\mathcal{B}e^2, \mathcal{B}b^2, \mathcal{B}|\mathbf{e}||\mathbf{b}|$ are very small compared to one. This is validated by the bound $\mathcal{B}(e^2+b^2) < 2.5\times 10^{-33}$ derived from Fig.1.

In order to make a more quantitative statement about this anisotropy let us consider the different possibilities arising from the velocity (137). To the leading order $\mathcal{B}|\mathbf{e}|^2$, $\mathcal{B}|\mathbf{b}|^2$, $\mathcal{B}|\mathbf{e}||\mathbf{b}|$ the magnitude of such velocity is

$$c(\hat{\mathbf{k}}) = 1 + 8\mathcal{B}\left(e^2+b^2\right) - 4\mathcal{B}\left(\left(\mathbf{b}\cdot\hat{\mathbf{k}}\right)^2 + \left(\mathbf{e}\cdot\hat{\mathbf{k}}\right)^2 - 2\hat{\mathbf{k}}\cdot\left(\mathbf{e}\times\mathbf{b}\right)\right), \tag{194}$$

which we take as the one-way light speed in the direction $\hat{\mathbf{k}}$, predicted by our model. The corresponding two-way light speed is then

$$c_{TW}(\hat{\mathbf{k}}) = \frac{1}{2}\left(c(\hat{\mathbf{k}}) + c(-\hat{\mathbf{k}})\right) = 1 + 8\mathcal{B}\left(e^2+b^2\right) - 4\mathcal{B}\left(\left(\mathbf{b}\cdot\hat{\mathbf{k}}\right)^2 + \left(\mathbf{e}\cdot\hat{\mathbf{k}}\right)^2\right). \tag{195}$$

A standard measure for the anisotropy of the speed of light is the comparison of the two-way velocities in perpendicular directions. We single out the vector $\left(\hat{\mathbf{e}}\times\hat{\mathbf{b}}\right) = \hat{\mathbf{n}}$ in our reference frame, which plays an analogous role to the relative

velocity among the ether frame and the laboratory frame in the Robertson-Mansouri-Sexl (RMS) type of analysis [34], [35]. The vectors $\hat{\mathbf{k}}$ and $\hat{\mathbf{n}}$ form a plane in which we define the vector $\hat{\mathbf{q}}$ perpendicular to $\hat{\mathbf{k}}$ given by

$$\hat{\mathbf{q}} = \hat{\mathbf{k}}\times\left(\hat{\mathbf{k}}\times\hat{\mathbf{n}}\right) = \left(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{k}} - \hat{\mathbf{n}}. \tag{196}$$

Next we calculate the two-way light speed along this perpendicular direction obtaining

$$c_{TW}(\hat{\mathbf{q}}) = 1 + 8\mathcal{B}\left(e^2+b^2\right) - 4\mathcal{B}\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\right)^2\left(b^2\left(\hat{\mathbf{b}}\cdot\hat{\mathbf{k}}\right)^2 + e^2\left(\hat{\mathbf{e}}\cdot\hat{\mathbf{k}}\right)^2\right). \tag{197}$$

An appropriate definition for the anisotropy of the speed of light in this model is

$$\frac{\Delta c}{c} \equiv \left|c_{TW}(\hat{\mathbf{k}}) - c_{TW}(\hat{\mathbf{q}})\right|, \tag{198}$$

$$\frac{\Delta c}{c} = \left|4\mathcal{B}\left(1-\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\right)^2\right)\left(b^2\left(\hat{\mathbf{b}}\cdot\hat{\mathbf{k}}\right)^2 + e^2\left(\hat{\mathbf{e}}\cdot\hat{\mathbf{k}}\right)^2\right)\right|. \tag{199}$$

From the last expression we obtain the bound

$$\frac{\Delta c}{c} < 4\mathcal{B}\sin^2\theta(\mathbf{b}^2+\mathbf{e}^2) < 4\mathcal{B}(\mathbf{b}^2+\mathbf{e}^2) < 10^{-32}, \tag{200}$$

according to Eq.(192), where $\theta$ is the angle between the vector $\hat{\mathbf{n}}$ and $\hat{\mathbf{k}}$. The above anisotropy measures the difference in the two-way speed of light propagating in perpendicular directions in a given reference frame. The standard Michelson-Morley type of analysis can be made by measuring first the time difference along two perpendicular trajectories $L_1$ (propagating with $c_{TW}(\hat{\mathbf{k}})$), and $L_2$ (propagating with $c_{TW}(\hat{\mathbf{q}})$)

$$\tau = \frac{L_2}{c_{TW}(\hat{\mathbf{q}})} - \frac{L_1}{c_{TW}(\hat{\mathbf{k}})}. \tag{201}$$

Subsequently a similar measurement is made by rotating the apparatus by 90 degrees

$$\tau' = \frac{L_2}{c_{TW}(-\hat{\mathbf{k}})} - \frac{L_1}{c_{TW}(\hat{\mathbf{q}})}. \tag{202}$$

Finally, the difference in optical paths

$$\Delta\tau = |\tau-\tau'| = 4\mathcal{B}\left(L_2+L_1\right)\sin^2\theta\left(b^2\left(\hat{\mathbf{b}}\cdot\hat{\mathbf{k}}\right)^2 + e^2\left(\hat{\mathbf{e}}\cdot\hat{\mathbf{k}}\right)^2\right) \tag{203}$$

measures the change in the interference pattern of the Michelson-Morley experiment, according to our model.

It is important to emphasize that the bound (200) is not related to the anisotropy in the speed of light induced by the passage from a preferred frame (where propagation is isotropic) to another frame moving with relative speed $\mathbf{v}$ and subjected to alternative methods of time synchronization. As such, it cannot be directly compared with the RMS type bounds appearing in the literature. The difference in two perpendicular two-way speed of light in the RMS case is [35]

$$\frac{\delta c}{c} = |c_{TW}(\phi+\pi/2) - c_{TW}(\phi)| = \left(\frac{v}{c}\right)^2\left|(\beta-\frac{1}{2}+\delta)\cos 2\phi\right|. \tag{204}$$

Here $\phi$ is the angle between the photon direction and the relative speed $\mathbf{v}$. Recent bounds upon the Mansouri-Sexl parameter $(\beta-\frac{1}{2}+\delta)$ [36] are of the order of $10^{-10}$, implying a RMS bound

$$\frac{\delta c}{c} \leq 10^{-16}, \tag{205}$$

with the relative speed between the earth and the CMB reference frame, where the cosmic background radiation looks isotropic, having the value $v = |\mathbf{v}| \simeq 300$ km/s.

Perhaps a more significant comparison of our results can be made with the work of Ref. [37], where the Euler model of nonlinear electrodynamics [38] is used to predict the following isotropic bound of the one-way speed of light

$$\frac{\tilde{\delta c}}{c} \leq 1.2\times 10^{-23}. \tag{206}$$

In our case this would correspond to averaging over all angles in (194), leading to

$$\frac{\tilde{\delta c}}{c} \leq 8\mathcal{B}\left(e^2 + b^2\right) \simeq 2 \times 10^{-32}. \tag{207}$$

Let us emphasize that in all our estimations we are assuming that the involved references frames (CMB, Sun based, earth based) are concordant, in such a way that first order quantities in the Lorentz invariance violation parameters remain of the same order in all of them.

Assuming that our background fields $\mathbf{e}$ and $\mathbf{b}$ might represent some galactic or intergalactic fields in the actual era, we obtain a very reasonable bound for the magnetic intergalactic field by assuming that the constant appearing in the action (38) corresponds to an energy density $\rho$

$$\rho = \frac{1}{4}\left[(1 - D^2\mathcal{B}\right] D^2 \simeq \frac{1}{2}\left(\mathbf{b}^2 - \mathbf{e}^2\right), \tag{208}$$

which we can associate with the cosmological constant, since it would represent a global property of the universe. The fact that this constant is positive favors $\left(\mathbf{b}^2 - \mathbf{e}^2\right) > 0$, so that one can perform a passive Lorentz transformation to a reference frame where $\mathbf{e} = \mathbf{0}$. Supposing further that this frame, which describes the intergalactic fields, is concordant with the Standard Inertial Reference Frame and taking the upper limit [39]

$$|\rho_\Lambda| < 10^{-48}\ (GeV)^4, \tag{209}$$

we obtain the bound

$$|\mathbf{b}| < 5 \times 10^{-5}\, Gauss, \tag{210}$$

which is consistent with observations of intergalactic magnetic fields. Let us recall that $1\ Gauss = 1.95 \times 10^{-20}\ (GeV)^2$. It is amusing to observe that this bound on $|\mathbf{b}|$ translates into the following condition for the mass scale $M_B$ introduced in Eq. (192)

$$M_{\mathcal{B}} > 1.4 \times 10^{-4}\ GeV. \tag{211}$$

Recalling that the lightest charged particle is the electron with $m_e = 5.1 \times 10^{-4}\ GeV$, we notice that the above bound is consistent with the QED Euler-Heisenberg type of non-linear corrections to the photon interaction, which scale as $1/m^4$, with $m$ being the mass of the charged particle contributing to the loop in the effective action [40].

## The Appendix

In this Appendix we extend the models in Refs [7, 8] to include all quartic gauge invariant fermionic interactions allowed by the Dirac algebra and look for their contribution to the effective gauge invariant electromagnetic action by integrating the fermion field. We heavily rely on the discussion and results of Ref.[25], which presents a complete and detailed description of the standard electromagnetic calculation. We do not attempt to cite the original papers, which references can be found in Ref.[25].

After integrating the massive gauge bosons in an underlying conventional theory [8], let us consider the effective Lagrangian

$$L_{eff} = \bar{\Psi}\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right)\Psi - \sum_{M,a}\frac{r_M}{2\Lambda^2}\left[\left(\bar{\Psi}M_a\Psi\right)\left(\bar{\Psi}M^a\Psi\right)\right]. \tag{212}$$

The index $M : S, V, T, PV, PS$ labels the tensorial objects that constitute the Dirac basis: Scalar, Vector, Tensor, Pseudovector and Pseudoscalar, respectively. The generic index $a$ labels the covariant (contravariant) components of each tensor class. More details are given in Table I. We follow the conventions of Ref.[22], with appropriate factors chosen in such a way that each current $\left(\bar{\Psi}M^a\Psi\right)$ is real.

Our goal is to estimate the contributions $W^{(2)}$ of the additional quartic fermionic couplings to the standard effective electromagnetic action $W^{(1)}_{EM}$. The latter is obtained by integrating the quadratic contribution of the fermions in Eq.(212) and it is given by

$$\begin{aligned}\langle 0_+|0_-\rangle^A_{(1)} &= \int\left[D\bar{\Psi}\right]\left[D\Psi\right]\,\exp\left[i\int d^4x\,\bar{\Psi}\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right)\Psi\right],\\ &= \det(\bar{\Psi}\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right)\Psi) \equiv \exp\left[iW^{(1)}_{EM}\right] = \exp\left[i\int d^4x\, L^{(1)}_{EM}\right].\end{aligned} \tag{213}$$

To this end we introduce the gauge invariant auxiliary fields $(C_M)_a$, via the couplings

$$L_{eff} = \bar{\Psi}\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right)\Psi + \sum_{M,a}\left[\frac{p_M}{2}(C_M)_a(C_M)^a - q_M(C_M)_a\left(\bar{\Psi}M^a\Psi\right)\right]. \tag{214}$$

The equations of motion for the auxiliary fields are (we do not use the Einstein convention for repeated indices unless explicitly stated)

$$(C_M)_a = \frac{q_M}{p_M}\left(\bar{\Psi}M_a\Psi\right), \tag{215}$$

which reproduce (212) when

$$\frac{r_M}{\Lambda^2} = \frac{q_M^2}{p_M}. \tag{216}$$

Next we integrate over the auxiliary fields following the standard steps which begin with the introduction of the currents $(j_M)^a$ together with the replacements

$$(C_M)_a \to \frac{\delta}{i\delta(j_M)^a}. \tag{217}$$

The total vacuum transition amplitude can then be written as

$$\langle 0_+|0_-\rangle^A = \left[\int [D\Psi]\left[D\bar{\Psi}\right][D(C_M)_a]\ \exp\left[i\int d^4x\ L_{eff}\right]\exp\left[i\sum_{M,a}\int d^4x\ (j_M)^a(C_M)_a\right]\right]_{(j_M)^a=0} \tag{218}$$

with

$$L_{eff} = \bar{\Psi}\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m - \sum_{M,a} q_M M^a\frac{\delta}{i\delta(j_M)^a}\right)\Psi + \sum_{M,a}\left[\frac{p_M}{2}(C_M)_a(C_M)^a\right]. \tag{219}$$

Now we can perform the Gaussian integral over the fields $(C_M)_a$, which includes the linear contribution of the currents, to obtain

$$\begin{aligned}\langle 0_+|0_-\rangle^A \ &=\ \int [D\Psi]\left[D\bar{\Psi}\right]\ \exp\left[i\int d^4x\ \bar{\Psi}\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m - \sum_{M,a} q_M M^a\frac{\delta}{i\delta(j_M)^a}\right)\Psi\right]\\ &\quad\times\exp\left[\int d^4x\sum_a\frac{i}{p_M}\ (j_M)^a(j_M)_a\right],\end{aligned} \tag{220}$$

where the limit $(j_M)_a \to 0$ is implicit. Notice that in order for the functional integral of the fields $(C_M)_a$ to be well defined in the usual Euclidean analytic continuation we must require that $p_a > 0$.

Using the standard formula (See for example Eq. (1.53) in Ref. [25])

$$F\left(\frac{\delta}{i\delta j}\right)\exp\left[\frac{i}{2}\int j\Delta j\right] = \left[\exp\left[\frac{i}{2}\int j\Delta j\right]\exp\left[-\frac{i}{2}\int\frac{\delta}{\delta X}\Delta\frac{\delta}{\delta X}\right]F(X)\right]_{X=\Delta j}, \tag{221}$$

where $\Delta(x-x')$ is proportional to $\delta(x-x')$ in each case, we can rewrite the above equation as

$$\begin{aligned}\langle 0_+|0_-\rangle^A \ &=\ \exp\left[-i\int d^4x\sum_{M,a}\frac{1}{2p_M}\ \frac{\delta}{\delta(X_M)^a}\frac{\delta}{\delta(X_M)_a}\right]\\ &\quad\times\int d\Psi d\bar{\Psi}\exp\left[i\int d^4x\ \bar{\Psi}\left(\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right) - \sum_{M,a} q_M M^a(X_M)_a\right)\Psi\right]_{(X_M)=0},\\ &= \left[\exp\left[-i\int d^4x\sum_{M,a}\frac{1}{2p_M}\ \frac{\delta}{\delta(X_M)^a}\frac{\delta}{\delta(X_M)_a}\right]\det\left(\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right) - \sum_{M,a} q_M M^a(X_M)_a\right)\right]_{X_M=0}.\end{aligned} \tag{222}$$

where we have already taken the limit $(j_M)^a = 0$. The next step is to rewrite

$$(i\gamma^\mu (\partial_\mu + ieA_\mu) - m) - \sum_{M,a} q_M M^a (X_M)_a = (i\gamma^\mu (\partial_\mu + ieA_\mu) - m) \left(1 - G_A \sum_{M,a} q_M M^a (X_M)_a\right), \tag{223}$$

where $G_A$ is the fermion Green function in the presence of the external electromagnetic field $A_\mu$, satisfying

$$(i\gamma^\mu (\partial_\mu + ieA_\mu) - m) G_A = 1. \tag{224}$$

Also we recall the expression

$$\det P = \exp [Tr \ln P], \tag{225}$$

where the trace is understood both in coordinate as well as in internal spaces. In this way we have

$$\begin{aligned} \langle 0_+ | 0_- \rangle^A &= \exp\left[iW^{(1)}_{EM}\right] \exp\left[-i \int d^4x \sum_{M,a} \frac{1}{2p_a} \frac{\delta}{\delta(X_M)^a} \frac{\delta}{\delta(X_M)_a}\right] \\ &\times \exp\left[Tr \ln\left(\left(1 - G_A \sum_{M,a} q_M M^a (X_M)_a\right)\right)\right]. \end{aligned} \tag{226}$$

To make an estimation analogous to the two-loop corrections $W^{(2)}_{EM}(A)$ to the effective action $W^{(1)}(A)$, which is already calculated in Ref. [25], we expand the third exponent up to terms quadratic in $(X_M)_a$. Calling the operator

$$G_A \sum_{M,a} q_M M^a (X_M)_a = \mathcal{G} = \sum_M \mathcal{G}_M \tag{227}$$

we are left with

$$\langle 0_+ | 0_- \rangle^A = \exp\left[iW^{(1)}_{EM}\right] \left[\exp\left[-i \int d^4x \sum_{M,a} \frac{1}{2p_M} \frac{\delta}{\delta(X_M)^a} \frac{\delta}{\delta(X_M)_a}\right] \exp\left[iTr\left(i\mathcal{G}\right)\right] \exp\left[\frac{i}{2} Tr\left(i\mathcal{G}^2\right)\right]\right]_{X_M=0}. \tag{228}$$

With the purpose of having a preliminary order of magnitude estimation of the corrections we next consider separately each of the terms $\mathcal{G}_M$ in the summation appearing in Eq. (227). In this way we are neglecting the contribution of the interference terms among different currents $j_M$, even though their contribution could also be calculated along the lines presented here.

In order to proceed we make use of another well-known identity

$$\begin{aligned} \exp\left[-\frac{i}{2} \int \frac{\delta}{\delta X} A \frac{\delta}{\delta X}\right] \exp\left[\frac{i}{2} \int XBX + i \int FX\right] &= \exp\left[+\frac{1}{2} Tr \ln (1 - BA)^{-1}\right] \\ &\times \exp\left[-\frac{i}{2} \int X\ B(1 - AB)^{-1}\ X\right] \\ &\times \exp\left[+i \int X(1 - BA)^{-1} F\right] \\ &\times \exp\left[+\frac{i}{2} \int FA(1 - BA)^{-1} F\right], \end{aligned} \tag{229}$$

for a fixed $M$. Next we identify the corresponding operators

$$\begin{aligned} (A_M)^b{}_c (x - x') &= \frac{1}{p_M} \delta^b_c \delta^4(x - x'), \quad F_{M_a} = i q_M G_A(x, x') M_a, \\ (B_M)^b{}_c (x, x') &= i\ q_M^2 G_A(x, x') M^b G_A(x', x) M_c\ , \end{aligned} \tag{230, 231}$$

where $\delta^b_c$ denotes the identity in the corresponding case. For example

$$M^a \to \gamma^\mu,\ \delta^b{}_c \to \delta^\mu_\nu\ ; \quad M^a \to \sigma^{\mu\nu},\ \delta^b{}_c \to \frac{1}{2}\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right). \tag{232}$$

| M | $M^c$ | $\theta_M = \sum_c tr(M^c M_c)$ | $\theta_{3M} = \sum_c tr(\sigma_3 M^c \sigma_3 M_c)$ |
|---|---|---|---|
| S | $I_{4\times 4}$ | 4 | 4 |
| V | $\gamma^\mu$ | 16 | 0 |
| T | $\sigma^{\mu\nu}$ | 48 | $-16$ |
| PV | $i\gamma^\mu\gamma_5$ | $-16$ | 0 |
| PS | $\gamma_5$ | $-4$ | $-4$ |

TABLE I: General notation and traces for the different contributions to $l_M$ in Eq.(238).

In the sequel we also restrict ourselves to the case of a constant (or slowly varying) external electromagnetic field. In this case $G_A(x,x')$ is independent of position so that the transformation properties under the Lorenz group imply that $F_{M_a} = 0$. (See for example Eq. (7.8) of Ref. [25]). This further simplifies Eq. (228) to

$$\langle 0_+|0_-\rangle_M^A = \exp\left[iW_{EM}^{(1)}\right]\exp\left[+\frac{1}{2}Tr\ln\left(1-B_M A_M\right)^{-1}\right] \simeq \exp\left[iW_{EM}^{(1)}\right]\exp\left[+\frac{1}{2}Tr\left(B_M A_M\right)\right] \tag{233}$$

This defines the sought correction

$$\exp\left[iW_M^{(2)}\right] = \exp\left[+\frac{1}{2}Tr\left(B_M A_M\right)\right], \quad M\ fixed \tag{234}$$

where

$$Tr\left(B_M A_M\right) = i\,\frac{r_M}{\Lambda^2}\sum_c\int d^4x\; tr\left(G_A(x,x)M^c G_A(x,x)M_c\right), \tag{235}$$

where $tr$ is the trace in the Dirac-matrices space. In other words we have

$$W_M^{(2)} = \frac{2r_M}{\Lambda^2}\sum_c\int d^4x\; tr\left(G_A(x,x)M^c G_A(x,x)M_c\right) \equiv \frac{r_M}{\Lambda^2}\int d^4x\; l_M. \tag{236}$$

In order to estimate each contribution we further consider the case where the external field is a constant magnetic $B$ field in the $z$-direction. We avoid the constant electric field case because of the inherent instability due to pair creation in the strong field regime. For the case under consideration we have [41]

$$G_A(x,x) = \frac{m}{16\pi^2}\int_0^\infty \frac{ds}{s^2}\,\exp\left[-ism^2\right]\frac{z}{\sin z}e^{i\sigma_3 z}, \quad s = eBs,\ \sigma_3 = i\gamma^1\gamma^2\ , \tag{237}$$

where $m$ is the mass of the integrated fermion. In this way, the general expression for the quantity $l_M$ defined in Eq. (236) is

$$\begin{aligned} l_M &= \left(\frac{m}{16\pi^2}\right)^2\int_0^\infty \frac{ds}{s^2}\frac{ds'}{s'^2}\exp\left[-ism^2\right]\exp\left[-is'm^2\right]\frac{z}{\sin z}\frac{z'}{\sin z'} \\ &\times\sum_c\left[\left(\cos z\cos z'\,tr\left(M^c M_c\right) - \sin z\sin z'\,tr\left(\sigma_3 M^c\sigma_3 M_c\right)\right) + \sin(z-z')tr(\sigma_3 M^c M_c)\right] \end{aligned} \tag{238}$$

The last term in the second line of Eq.(238) vanishes because $\sum_c M^c M_c$ is always a multiple of the identity and $tr(\sigma_3) = 0$. Let us introduce the notation

$$\theta_M = tr\left(\sum_c M^c M_c\right), \quad \theta_{3M} = tr\left(\sum_c \sigma_3 M^c\sigma_3 M_c\right), \tag{239}$$

which values are given in Table I. Also it is convenient to define

$$I_1 = \int_0^\infty \frac{ds}{s^2}\,\exp\left[-ism^2\right]\frac{z\cos z}{\sin z}, \quad I_2 = \int_0^\infty\frac{ds}{s^2}\,\exp\left[-ism^2\right]z = eB\int_0^\infty\frac{ds}{s}\,\exp\left[-ism^2\right]. \tag{240}$$

In terms of the above quantities we have

$$l_M = \left(\frac{m}{16\pi^2}\right)^2\left[\theta_M I_1^2\ - \theta_{3M}I_2^2\right]. \tag{241}$$

The second contribution, with infinite coefficient, behaves as $B^2$ so that we consider it as part of the renormalization procedure that we demand of the low energy limit of the effective potential

$$\lim_{B\to 0} V_{eff}(B) = -\rho B^2, \qquad \rho > 0. \tag{242}$$

In this way we consider the full expression for the contribution due to the additional currents to be

$$L^{(2)} = \sum_M L_M^{(2)}, \qquad L_M^{(2)} = \left(\frac{m}{16\pi^2}\right)^2 \frac{r_M}{\Lambda^2}\, \theta_M\, I_1^2. \tag{243}$$

The effective Lagrangians in (243) lead to the effective potentials

$$V_M^{(2)} = -L_M^{(2)} = -\left(\frac{m}{16\pi^2}\right)^2 \frac{r_M}{\Lambda^2}\, \theta_M\, I_1^2. \tag{244}$$

The main point to be made is that each contribution to the effective potential arising from the subspace $M$ contributes with an specific sign, as can be seen from Table 1. In these way, as it will be argued in the sequel, it is possible to have a net positive contribution for the total effective potential in the limit $B/B_0 \to \infty$, where $B_0 = m^2/e$ is the critical value of the magnetic field. Let us emphasize that $V_M^{(2)}$ is still not well defined because the integral $I_1$ diverges. This means that we should regularize it and further implement a renormalization prescription for the full effective Lagrangian

$$L_{eff} = \alpha B^2 + L_{EM}^{(1)}(B) + L_{EM}^{(2)}(B) + L^{(2)}(B). \tag{245}$$

Here $L_{EM}^{(2)}(B)$ refers to the two-loop standard electromagnetic correction. Our renormalization conditions will be

$$\lim_{B\to 0} L_{eff} = \rho B^2, \qquad \rho > 0, \tag{246}$$

in such a way that the effective potential is a decreasing function in the vicinity of $B = 0$. The final normalization to the value of $L_{eff} = -b^2/2, b \to 0$, will be imposed at the end of the procedure, after we expand the magnetic field around the expected stable minimum $B_{Min}$ of $V_{eff}$, such that

$$B = B_{Min} + b \tag{247}$$

and we are left with the physical component $b$. The renormalization procedure should be analogous to the one carried out in Ref. [25] for the contribution $L_{EM}^{(2)}(A)$ and it is rather involved due to the complicated field dependence of the regularized version of the divergent integrals. This calculation is out of the scope of these preliminary estimations.

Thus we will only consider a particular regularization of $I_1$ to proceed with the comparison of the different high-intensity field contributions to the effective potential. We take

$$I_1 \to I_1 = \int_0^\infty \frac{ds}{s^2}\, \exp\left[-ism^2\right] \left[\frac{z\cos z}{\sin z} - 1\right], \tag{248}$$

which produces a zero contribution when $B \to 0$. This integral can be readily calculated from the first order electromagnetic correction of the effective action given by [25]

$$L_{EM}^{(1)} = \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} \exp\left[-ism^2\right] \left[\frac{z\cos z}{\sin z} + \frac{1}{3}z - 1\right]. \tag{249}$$

The result is

$$I_1 = i\left[8\pi^2 \frac{\partial L_{EM}^{(1)}}{\partial m^2} + \frac{eB^2}{3m^2}\right], \tag{250}$$

which yields our final result

$$V_M^{(2)} = \left(\frac{m}{16\pi^2}\right)^2 \frac{r_M}{\Lambda^2}\, \theta_M \left[8\pi^2 \frac{\partial L_{EM}^{(1)}}{\partial m^2} + \frac{eB^2}{3m^2}\right]^2. \tag{251}$$

In order to produce the required results in the low (high)-intensity field approximations, we list here the behavior of the relevant terms obtained from Ref. [25]. For $B \rightarrow \infty$ we have

$$V_{EM}^{(1)} \rightarrow -\frac{e^2}{24\pi^2} B^2 \ln\left(\frac{eB}{m^2}\right), \tag{252}$$

$$V_{EM}^{(2)} \rightarrow -\frac{e^4}{128\pi^4} B^2 \ln\left(\frac{eB}{m^2}\right), \tag{253}$$

$$8\pi^2 \frac{\partial L_{EM}^{(1)}}{\partial m^2} + \frac{eB^2}{3m^2} \rightarrow eB \ln\left(\frac{2eB}{m^2}\right). \tag{254}$$

The limit $B \rightarrow 0$ produces

$$V_{EM}^{(1)} \rightarrow -\frac{e^4}{360\pi^2 m^4} B^4, \tag{255}$$

$$8\pi^2 \frac{\partial L_{EM}^{(1)}}{\partial m^2} + \frac{eB^2}{3m^2} \rightarrow \left(\frac{eB^2}{3m^2} + \frac{1}{18}\frac{e^2 B^4}{m^4}\right). \tag{256}$$

In this way, the final contributions of a given current $j_M$ to the effective potential are

$$B \rightarrow \infty: \qquad V_M^{(2)} = \left(\frac{1}{16\pi^2}\right)^2 \left(\frac{m}{\Lambda}\right)^2 r_M\, \theta_M\, e^2 B^2 \left[\ln\left(\frac{2eB}{m^2}\right)\right]^2, \tag{257}$$

$$B \rightarrow 0: \qquad V_M^{(2)} = \left(\frac{1}{16\pi^2}\right)^2 \left(\frac{m}{\Lambda}\right)^2 r_M\, \theta_M \left(\frac{eB^2}{3m^2} + \frac{1}{18}\frac{e^2 B^4}{m^4}\right)^2. \tag{258}$$

The total contribution from the currents $j_M$ is

$$V^{(2)} = \sum_M V_M^{(2)}. \tag{259}$$

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